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Γ -limits of functionals determined by their minima

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Γ -LIMITS OF FUNCTIONALS DETERMINED BY THEIR INFIMA

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In memoriam Jean-Jacques Moreau

ABSTRACT. We study the integral representation of Γ -limits of p -coercive integral functionals of the calculus of variations in the spirit of [DMM86a]. We use infima of local Dirichlet problems to characterize the limit integrands. Applications to homogenization and relaxation are given.

1. INTRODUCTION

Let $m, d \geq 1$ be two integers. Let $\Omega \subset \mathbb{R}^d$ be a nonempty bounded open set with Lipschitz boundary. Let $\mathcal{O}(\Omega)$ be the class of all open subsets of Ω . We consider a family of functionals $\mathcal{F} := \{F_\varepsilon\}_{\varepsilon \in]0,1]}$ with $F_\varepsilon : W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \rightarrow [0, \infty]$. We set conditions in order that each functional of the family \mathcal{F} can be considered as a p -coercive integral functional of the calculus of variations (see the ‘‘global’’ conditions (C_1) , (C_2) , (C_3) in Sect. 2). We are interested in the integral representation of the $\Gamma(L^p)$ -limit of \mathcal{F} . This is an important problem in the field of Γ -convergence theory (see for instance [DG79]).

Our goal is to study the conditions of the integral representation of $\Gamma(L^p)$ -limit by using the infima of local Dirichlet problems associated to \mathcal{F} as in [DMM86a, BFM98, BFLM02]. More precisely, we consider the behavior of

$$m_\varepsilon(u; O) := \inf \left\{ F_\varepsilon(v; O) : v \in u + W_0^{1,p}(O; \mathbb{R}^m) \right\}$$

in order to find the conditions for the integral representation (see also [DMM86b, DMM86c, Mod86]). We propose three ‘‘local’’ conditions (see (H_1) , (H_2) and (H_3) in Sect. 2) related to the local behavior of m_ε which allows to prove $\Gamma(L^p)$ -convergence of the family $\mathcal{F}(\cdot; O)$ with integral representation of the $\Gamma(L^p)$ -limit $\mathcal{F}_0(\cdot; O)$

$$\mathcal{F}_0(u; O) = \int_O L_0(x, u(x), \nabla u(x)) dx$$

where $u \in M_{\mathcal{F}}(O)$ (see Definition (2.1) for $M_{\mathcal{F}}(O)$) and

$$L_0(x, u(x), \nabla u(x)) = \lim_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_x; Q_\rho(x))}{\rho^d} \quad \text{with } u_x(\cdot) = u(x) + \nabla u(x)(\cdot - x).$$

The main difficulty is to obtain an upper bound under integral form for the $\Gamma(L^p)$ - $\overline{\lim}$. More precisely, we show, in Sect. 3 together with Sect. 4, that the Vitali envelope (which is an envelope of Carathéodory type where the arbitrary coverings are replaced by Vitali coverings) $V_+(u; \cdot)$ of the set function $\mathcal{O}(\Omega) \ni V \mapsto \overline{\lim}_{\varepsilon \rightarrow 0} m_\varepsilon(u; V)$ when $u \in M_{\mathcal{F}}(O)$ satisfies

$$\Gamma(L^p)\text{-}\overline{\lim}_{\varepsilon \rightarrow 0} F_\varepsilon(u; O) \leq V_+(u; O) = \int_O \liminf_{\rho \rightarrow 0} \left\{ \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u; Q)}{\lambda(Q)} : x \in Q \in \mathcal{Q}_o(O), \text{diam}(Q) \leq \rho \right\} dx.$$

The Vitali envelope of a set function in connection with the integral representation of $\Gamma(L^p)$ -limits was introduced in [BFM98] (see also [BB00]). This path has the advantage to avoid any approximations of Sobolev functions by regular ones. It allows, when we assume p -growth conditions, to give general results for $\Gamma(L^p)$ -limit and in particular to give a general point of view in homogenization and relaxation problems for Borel measurable integrands $L(x, v, \xi)$ (see Sect. 5).

Plan of the paper. Sect. 2 presents the main assumptions (‘‘global and local’’ conditions) and the statement of the general results (see Theorem 2.1 and Theorem 2.2). Theorem 2.2 is an integral representation result of $\Gamma(L^p)$ -limit, it is a consequence of local conditions ((H_1) , (H_2) and (H_3)) and Theorem 2.1. In Sect. 3 we state and prove an integral representation for the Vitali envelope of arbitrary nonnegative set functions. In Sect. 4 we give the proof of Theorem 2.1 and some other related results. Finally in Sect. 5 we give a general $\Gamma(L^p)$ -convergence result in the p -growth case Theorem 5.1, which can be seen as an

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extension in a nonconvex (an vectorial) case of Theorem IV in [DMM86a, p. 265]. In fact, we show how to verify the local conditions (H₂) and (H₃) when we deal with p -growth, the technics we use are inspired by [BFM98]. In Subsect. 5.2 as an application of Theorem 5.1 we consider a general point of view of the homogenization of functional integral of the calculus of variations. In Subsect. 5.3 we give an extension of the Acerbi-Fusco-Dacorogna relaxation theorem when the integrand is assumed Borel measurable only.

2. MAIN RESULTS

2.1. General framework. Fix $\alpha > 0$ and $p \in]1, \infty[$. We denote by $\mathcal{I}(p, \alpha)$ the set of functionals $F : W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \rightarrow [0, \infty]$ satisfying:

(C₁) for every $O \in \mathcal{O}(\Omega)$ and every $u \in \text{dom}F(\cdot; O)$ we have

$$F(u; O) \geq \alpha \|\nabla u\|_{L^p(O; \mathbb{R}^m)}^p;$$

(C₂) for every $u \in \text{dom}F(\cdot; \Omega)$ the set function $F(u; \cdot)$ is the trace on $\mathcal{O}(\Omega)$ of a Borel measure absolutely continuous with respect to the Lebesgue measure λ on Ω ;

(C₃) for every $O \in \mathcal{O}(\Omega)$ the functional $F(\cdot; O)$ is local, i.e., if $u = v$ a.e. in O then $F(u; O) = F(v; O)$ for all $u, v \in \text{dom}F(\cdot; O)$.

Consider a family $\mathcal{F} := \{F_\varepsilon\}_{\varepsilon \in]0,1]}$ of functionals $F_\varepsilon : W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \rightarrow [0, \infty]$. For each $O \in \mathcal{O}(\Omega)$ and each $u \in L^p(\Omega; \mathbb{R}^m)$ we set

$$\begin{aligned} \mathcal{F}_-(u; O) &:= \inf \left\{ \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon; O) : u_\varepsilon \rightarrow u \text{ in } L^p(\Omega; \mathbb{R}^m) \right\}; \\ \mathcal{F}_+(u; O) &:= \inf \left\{ \overline{\lim}_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon; O) : u_\varepsilon \rightarrow u \text{ in } L^p(\Omega; \mathbb{R}^m) \right\}. \end{aligned}$$

The functional $\mathcal{F}_-(\cdot; O)$ (resp. $\mathcal{F}_+(\cdot; O)$) is the $\Gamma(L^p)$ - $\underline{\lim}_{\varepsilon \rightarrow 0}$ (resp. $\Gamma(L^p)$ - $\overline{\lim}_{\varepsilon \rightarrow 0}$) of the family $\mathcal{F}(\cdot; O) = \{F_\varepsilon(\cdot; O)\}_\varepsilon$. If $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ and $\mathcal{F}_+(u; O) = \mathcal{F}_-(u; O)$ then we say that $\mathcal{F}(\cdot; O)$ $\Gamma(L^p)$ -converges at u to the $\Gamma(L^p)$ -limit $\mathcal{F}_0(u; O) := \mathcal{F}_+(u; O) = \mathcal{F}_-(u; O)$.

We associate to $\mathcal{F} = \{F_\varepsilon\}_{\varepsilon \in]0,1]}$ a family of local Dirichlet problems $\{m_\varepsilon\}_{\varepsilon \in]0,1]}$, $m_\varepsilon : W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \rightarrow [0, \infty]$ defined by

$$m_\varepsilon(u; O) := \inf \{F_\varepsilon(v; O) : W^{1,p}(\Omega; \mathbb{R}^m) \ni v = u \text{ in } \Omega \setminus O\}.$$

Note that we can write

$$m_\varepsilon(u; O) = \inf \{F_\varepsilon(v; O) : v \in u + W_0^{1,p}(O; \mathbb{R}^m)\}$$

since $u + W_0^{1,p}(O; \mathbb{R}^m) = \{v \in W^{1,p}(\Omega; \mathbb{R}^m) : v - u = 0 \text{ in } \Omega \setminus O\}$ (see [AH96, p. 234, Theorem 9.1.3]).

Remark 2.1. The functional $m_\varepsilon(\cdot; O)$ can be seen as the ‘‘quotient functional’’ $\widetilde{F}_\varepsilon(\cdot; O)$ defined on the quotient space of $W^{1,p}(\Omega; \mathbb{R}^m)$ by $W_0^{1,p}(O; \mathbb{R}^m)$, i.e.,

$$\widetilde{F}_\varepsilon(\cdot; O) : W^{1,p}(\Omega; \mathbb{R}^m)/W_0^{1,p}(O; \mathbb{R}^m) \rightarrow [0, \infty]$$

$$\text{with } \widetilde{F}_\varepsilon([u]; O) := \inf_{v \in [u]} F_\varepsilon(v; O) = m_\varepsilon(u; O)$$

where $[u] = u + W_0^{1,p}(O; \mathbb{R}^m)$ is the equivalent class of u .

2.2. A general $\Gamma(L^p)$ -convergence theorem. We denote by λ the Lebesgue measure on Ω . For each $O \in \mathcal{O}(\Omega)$ we denote by $\mathfrak{A}_\lambda(O)$ the space of nonnegative *finite* Borel measures on O which are absolutely continuous with respect to the Lebesgue measure $\lambda|_O$ on O .

Let us introduce the *M-sets associated to \mathcal{F}* : for each $O \in \mathcal{O}(\Omega)$ we set

$$(2.1) \quad \text{M}_{\mathcal{F}}(O) := \left\{ u \in W^{1,p}(\Omega; \mathbb{R}^m) : \exists \mu_u \in \mathfrak{A}_\lambda(O) \quad \sup_{\varepsilon > 0} m_\varepsilon(u; \cdot) \leq \mu_u(\cdot) \text{ on } O \right\}.$$

- We assume that all the affine maps, i.e., functions of the form $u(x) = v + \zeta x$ with $(x, v, \zeta) \in \Omega \times \mathbb{R}^m \times \mathbb{M}^{m \times d}$, belong to $\text{M}_{\mathcal{F}}(O)$.

We will see in Theorem 2.2 that the M-set is the set where an integral representation of the $\Gamma(L^p)$ -limit is possible.

To the family \mathcal{F} we associate $m_+ : W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \rightarrow [0, \infty]$ defined by

$$m_+(u; O) := \overline{\lim}_{\varepsilon \rightarrow 0} m_\varepsilon(u; O).$$

The following result provides bounds in integral forms of both $\Gamma(L^p)\text{-}\underline{\lim}_{\varepsilon \rightarrow 0}$ and $\Gamma(L^p)\text{-}\overline{\lim}_{\varepsilon \rightarrow 0}$ of a family $\mathcal{F} = \{F_\varepsilon\}_{\varepsilon \in]0,1[} \subset \mathcal{I}(p, \alpha)$, i.e., satisfying (C₁), (C₂) and (C₃).

Theorem 2.1. *Let $\mathcal{F} = \{F_\varepsilon\}_{\varepsilon \in]0,1[} \subset \mathcal{I}(p, \alpha)$ and let $(u, O) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega)$.*

(i) *If $u \in M_{\mathcal{F}}(O)$ then*

$$\mathcal{F}_+(u; O) \leq \mathcal{F}_+^{\mathfrak{D}}(u; O) \leq \int_O \liminf_{\rho \rightarrow 0} \left\{ \frac{m_+(u; Q)}{\lambda(Q)} : x \in Q \in \mathcal{Q}_o(O), \text{diam}(Q) \leq \rho \right\} dx$$

where

$$\mathcal{F}_+^{\mathfrak{D}}(u; O) := \inf \left\{ \overline{\lim}_{\varepsilon \rightarrow 0} F_\varepsilon(v_\varepsilon; O) : W_0^{1,p}(O; \mathbb{R}^m) + u \ni v_\varepsilon \rightarrow u \text{ in } L^p(\Omega; \mathbb{R}^m) \right\};$$

(ii) *There exists $\{u_{\varepsilon_n}\}_n \subset W^{1,p}(\Omega; \mathbb{R}^m)$ with $\sup_n F_{\varepsilon_n}(u_{\varepsilon_n}; O) < \infty$ such that $u_{\varepsilon_n} \rightarrow u$ in $L^p(\Omega; \mathbb{R}^m)$ as $n \rightarrow \infty$ and*

$$\mathcal{F}_-^{\mathfrak{D}}(u; O) \geq \mathcal{F}_-(u; O) \geq \int_O \lim_{\rho \rightarrow 0} \lim_{n \rightarrow \infty} \frac{F_{\varepsilon_n}(u_{\varepsilon_n}; Q_\rho(x))}{\rho^d} dx$$

where

$$\mathcal{F}_-^{\mathfrak{D}}(u; O) := \inf \left\{ \underline{\lim}_{\varepsilon \rightarrow 0} F_\varepsilon(v_\varepsilon; O) : W_0^{1,p}(O; \mathbb{R}^m) + u \ni v_\varepsilon \rightarrow u \text{ in } L^p(\Omega; \mathbb{R}^m) \right\}.$$

Remark 2.2. If $u \in M_{\mathcal{F}}(O)$ then there exists $\mu_u \in \mathfrak{A}_\lambda(O)$ such that $\sup_{\varepsilon > 0} m_\varepsilon(u; \cdot) \leq \mu_u(\cdot)$ on O . Therefore we have

$$m_+(u; \cdot) \leq \mu_u(\cdot) \text{ on } O.$$

Taking account of Theorem 2.1 (i) we deduce that $\mathcal{F}_+(u; O) < \infty$, which means that

$$M_{\mathcal{F}}(O) \subset \text{dom} \mathcal{F}_+^{\mathfrak{D}}(\cdot; O) := \{u \in W^{1,p}(\Omega; \mathbb{R}^m) : \mathcal{F}_+^{\mathfrak{D}}(u; O) < \infty\} \subset \text{dom} \mathcal{F}_+(\cdot; O).$$

To the family $\mathcal{F} = \{F_\varepsilon\}_{\varepsilon \in]0,1[}$ we associate $m_- : W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \rightarrow [0, \infty]$ defined by

$$m_-(u; O) := \underline{\lim}_{\varepsilon \rightarrow 0} m_\varepsilon(u; O).$$

Let $O \in \mathcal{O}(\Omega)$ and let $u \in W^{1,p}(O; \mathbb{R}^m)$. We denote the affine tangent map of u at $x \in O$ by

$$u_x(\cdot) := u(x) + \nabla u(x)(\cdot - x).$$

Consider the following local inequalities for $u \in M_{\mathcal{F}}(O)$:

$$(H_1) \quad \underline{\lim}_{\rho \rightarrow 0} \frac{m_-(u_x; Q_\rho(x))}{\rho^d} \geq \underline{\lim}_{\rho \rightarrow 0} \frac{m_+(u_x; Q_\rho(x))}{\rho^d} \quad \text{a.e. in } O;$$

$$(H_2) \quad \underline{\lim}_{\rho \rightarrow 0} \underline{\lim}_{\varepsilon \rightarrow 0} \frac{F_\varepsilon(u_\varepsilon; Q_\rho(x))}{\rho^d} \geq \underline{\lim}_{\rho \rightarrow 0} \frac{m_-(u_x; Q_\rho(x))}{\rho^d} \quad \text{a.e. in } O \quad \text{for all } \{u_\varepsilon\}_\varepsilon \subset W^{1,p}(\Omega; \mathbb{R}^m) \text{ such that } u_\varepsilon \rightarrow u \text{ in } L^p(\Omega; \mathbb{R}^m) \text{ and } \sup_\varepsilon F_\varepsilon(u_\varepsilon; O) < \infty;$$

$$(H_3) \quad \underline{\lim}_{\rho \rightarrow 0} \frac{m_+(u_x; Q_\rho(x))}{\rho^d} \geq \underline{\lim}_{\rho \rightarrow 0} \frac{m_+(u; Q_\rho(x))}{\rho^d} \quad \text{a.e. in } O.$$

Remark 2.3. We make some remarks on the previous inequalities.

- (i) Similar condition to (H_1) , related to the integral representation of the $\Gamma(L^p)$ -limit of functionals of the calculus of variations, is already known when p -polynomial growth (and convexity conditions) is assumed see [Mas06, p. 451]. In fact, since $m_+(u; O) \geq m_-(u; O)$ for all $(u, O) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega)$, the condition (H_1) can be rewritten as follows

$$\overline{\lim}_{\rho \rightarrow 0} \frac{m_-(u_x; Q_\rho(x))}{\rho^d} = \overline{\lim}_{\rho \rightarrow 0} \frac{m_+(u_x; Q_\rho(x))}{\rho^d} \quad \text{a.e. in } O.$$

We see that a sufficient condition for (H_1) to hold is

$$\lim_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_x; Q)}{\lambda(Q)} \quad \text{exists for all open cube } Q \subset O \text{ and all } x \in O.$$

- (ii) The condition (H_2) (resp. (H_3)) can be seen as a “local” $\Gamma(L^p)$ - $\underline{\lim}$ (resp. $\Gamma(L^p)$ - $\overline{\lim}$) inequality. To verify inequality (H_3) (resp. (H_2)) we need to replace u (resp. a sequence $\{u_\varepsilon\}_\varepsilon$ converging in L^p to u and satisfying $\sup_\varepsilon F_\varepsilon(u_\varepsilon; O) < \infty$) by the affine tangent map u_x in the localization of m_ε on “small” cubes $Q_\rho(x)$. This can be performed, for instance, by using growth conditions see Sect. 5.
- (iii) If $\{F_\varepsilon = F\}_\varepsilon = \mathcal{F}$ is constant with respect to ε then (H_1) is always satisfied.

The following lemma is used in the proof of Theorem 2.2 and its proof is given in Sect. 4.

Lemma 2.1. *Let $\mathcal{F} = \{F_\varepsilon\}_{\varepsilon \in]0,1]} \subset \mathcal{I}(p, \alpha)$. Let $O \in \mathcal{O}(\Omega)$ and let $u \in M_{\mathcal{F}}(O)$. If (H_1) , (H_2) and (H_3) hold then the following limit $\lim_{\rho \rightarrow 0} \frac{m_+(u_x; Q_\rho(x))}{\rho^d}$ exists almost everywhere in O . Moreover the function $O \ni x \mapsto \lim_{\rho \rightarrow 0} \frac{m_+(u_x; Q_\rho(x))}{\rho^d}$ is measurable and satisfies*

$$\lim_{\rho \rightarrow 0} \frac{m_+(u_x; Q_\rho(x))}{\rho^d} = \overline{\lim}_{\rho \rightarrow 0} \frac{m_-(u_x; Q_\rho(x))}{\rho^d} = \lim_{\rho \rightarrow 0} \frac{m_+(u; Q_\rho(x))}{\rho^d} \quad \text{a.e. in } O.$$

Here is the general $\Gamma(L^p)$ -convergence theorem which shows that under the local inequalities (H_1) , (H_2) and (H_3) the family $\mathcal{F}(\cdot; O)$ $\Gamma(L^p)$ -converges to an integral functional of the calculus of variations at every $u \in M_{\mathcal{F}}(O)$. In Sect. 5 we give applications to homogenization and relaxation of this result. When $\mathcal{F}_+^{\mathfrak{D}} = \mathcal{F}_-^{\mathfrak{D}}$ we denote by $\mathcal{F}_0^{\mathfrak{D}} = \mathcal{F}_+^{\mathfrak{D}} = \mathcal{F}_-^{\mathfrak{D}}$ the common value.

Theorem 2.2. *Let $\mathcal{F} = \{F_\varepsilon\}_{\varepsilon \in]0,1]} \subset \mathcal{I}(p, \alpha)$. Let $O \in \mathcal{O}(\Omega)$ and let $u \in M_{\mathcal{F}}(O)$. If (H_1) , (H_2) and (H_3) hold then the family of functionals $\mathcal{F}(\cdot; O)$ $\Gamma(L^p)$ -converges at u to*

$$(2.2) \quad \mathcal{F}_0(u; O) = \mathcal{F}_0^{\mathfrak{D}}(u; O) = \int_O \lim_{\rho \rightarrow 0} \frac{m_+(u_x; Q_\rho(x))}{\rho^d} dx.$$

Moreover, we have for almost all $x \in O$

$$(2.3) \quad \lim_{\rho \rightarrow 0} \frac{m_+(u_x; Q_\rho(x))}{\rho^d} = \overline{\lim}_{\rho \rightarrow 0} \frac{m_-(u_x; Q_\rho(x))}{\rho^d}.$$

Proof. Let $O \in \mathcal{O}(\Omega)$ and let $u \in M_{\mathcal{F}}(O)$. From Theorem 2.1 (ii), there exists $\{u_\varepsilon\}_\varepsilon \subset W^{1,p}(\Omega; \mathbb{R}^m)$ with $\sup_\varepsilon F_\varepsilon(u_\varepsilon; O) < \infty$ such that $u_\varepsilon \rightarrow u$ in $L^p(\Omega; \mathbb{R}^m)$ as $\varepsilon \rightarrow 0$ and

$$\mathcal{F}_-^{\mathfrak{D}}(u; O) \geq \mathcal{F}_-(u; O) \geq \int_O \lim_{\rho \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{F_\varepsilon(u_\varepsilon; Q_\rho(x))}{\rho^d} dx$$

Using the local inequalities (H_1) , (H_2) and (H_3) , Lemma 2.1 together with Theorem 2.1 we have

$$\begin{aligned} \mathcal{F}_+(u; O) &\leq \mathcal{F}_+^{\mathfrak{D}}(u; O) \leq \int_O \liminf_{\rho \rightarrow 0} \left\{ \frac{m_+(u; Q)}{\lambda(Q)} : x \in Q \in \mathcal{Q}_o(O), \text{diam}(Q) \leq \rho \right\} dx \\ &\leq \int_O \lim_{\rho \rightarrow 0} \frac{m_+(u; Q_\rho(x))}{\rho^d} dx \\ &\leq \int_O \lim_{\rho \rightarrow 0} \frac{m_+(u_x; Q_\rho(x))}{\rho^d} dx \\ &= \int_O \overline{\lim}_{\rho \rightarrow 0} \frac{m_-(u_x; Q_\rho(x))}{\rho^d} dx \\ &\leq \int_O \lim_{\rho \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{F_\varepsilon(u_\varepsilon; Q_\rho(x))}{\rho^d} dx \\ &\leq \mathcal{F}_-(u; O) \leq \mathcal{F}_-^{\mathfrak{D}}(u; O). \end{aligned}$$

Thus (2.2) holds. The equality (2.3) is a consequence of Lemma 2.1. \blacksquare

2.3. On the condition (H₁). The following result shows that the condition (H₁) holds whenever $\mathcal{F}_-^{\mathfrak{D}}$ and $\mathcal{F}_+^{\mathfrak{D}}$ are equal.

Proposition 2.1. *If for every cube $Q \in \mathcal{O}(\Omega)$ and every $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ we have for every $x \in \Omega$*

$$\inf \left\{ \mathcal{F}_-^{\mathfrak{D}}(v; Q) : v \in u_x + W_0^{1,p}(Q; \mathbb{R}^m) \right\} = \inf \left\{ \mathcal{F}_+^{\mathfrak{D}}(v; Q) : v \in u_x + W_0^{1,p}(Q; \mathbb{R}^m) \right\}$$

then (H₁) holds.

Proof. Let $Q \in \mathcal{O}(\Omega)$ be a cube, $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ and $x \in \Omega$. We set

$$l(u_x; Q) := \inf \left\{ \mathcal{F}_-^{\mathfrak{D}}(v; Q) : v \in u_x + W_0^{1,p}(Q; \mathbb{R}^m) \right\} = \inf \left\{ \mathcal{F}_+^{\mathfrak{D}}(v; Q) : v \in u_x + W_0^{1,p}(Q; \mathbb{R}^m) \right\}.$$

Let $\{\varepsilon_n\}_n \subset]0, 1]$ be a sequence such that $\lim_{n \rightarrow \infty} \varepsilon_n = 0$. There exists a sequence $\{v_n\}_n \subset u_x + W_0^{1,p}(Q; \mathbb{R}^m)$ such that

$$\liminf_{n \rightarrow \infty} F_{\varepsilon_n}(v_n; Q) = \liminf_{n \rightarrow \infty} m_{\varepsilon_n}(u_x; Q) \leq \mu_{u_x}(Q) < \infty.$$

since $u_x \in M_{\mathcal{F}}(Q)$. There exists a subsequence such that

$$\liminf_{n \rightarrow \infty} m_{\varepsilon_n}(u; Q) = \liminf_{n \rightarrow \infty} F_{\varepsilon_{\sigma(n)}}(v_{\sigma(n)}; Q) \text{ and } v_{\sigma(n)} \rightarrow v_{\infty} \in u + W_0^{1,p}(Q; \mathbb{R}^m) \text{ in } L^p(\Omega; \mathbb{R}^m)$$

since p -coercivity. Therefore

$$\liminf_{n \rightarrow \infty} m_{\varepsilon_n}(u_x; Q) \geq \mathcal{F}_-^{\mathfrak{D}}(v_{\infty}; Q) \geq l(u_x; Q).$$

For any $v \in u_x + W_0^{1,p}(Q; \mathbb{R}^m)$ we have

$$\overline{\lim}_{n \rightarrow \infty} m_{\varepsilon_n}(u_x; Q) \leq \mathcal{F}_+^{\mathfrak{D}}(v; Q)$$

that means

$$\overline{\lim}_{n \rightarrow \infty} m_{\varepsilon_n}(u_x; Q) \leq l(u_x; Q).$$

We deduce that the following limit exists for every cube $Q \in \mathcal{O}(\Omega)$ and every $u \in W^{1,p}(\Omega; \mathbb{R}^m)$

$$l(u_x; Q) = \lim_{n \rightarrow \infty} m_{\varepsilon_n}(u_x; Q) = \lim_{\varepsilon \rightarrow 0} m_{\varepsilon}(u_x; Q) = \overline{\lim}_{\varepsilon \rightarrow 0} m_{\varepsilon}(u_x; Q).$$

It follows that

$$\overline{\lim}_{\rho \rightarrow 0} \lim_{n \rightarrow \infty} \frac{m_{\varepsilon_n}(u_x; Q_{\rho}(x))}{\rho^d} = \overline{\lim}_{\rho \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{m_{\varepsilon}(u_x; Q_{\rho}(x))}{\rho^d} = \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_{\varepsilon}(u_x; Q_{\rho}(x))}{\rho^d}. \quad \blacksquare$$

Remark 2.4. Let $O \in \mathcal{O}(\Omega)$ and $u \in M_{\mathcal{F}}(O)$. Assume that (H₂) and (H₃) hold. Then (H₁) holds if and only if $\mathcal{F}_+^{\mathfrak{D}}(u; O) = \mathcal{F}_-^{\mathfrak{D}}(u; O)$.

2.4. The relaxation case. We examine the particular case of a constant family with respect to the parameter $\mathcal{F} = \{F_{\varepsilon} = F\}_{\varepsilon} \subset \mathcal{I}(p, \alpha)$. We set for every $(u, O) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega)$

$$\begin{aligned} \mathcal{F}_0(u; O) &:= \inf \left\{ \liminf_{\varepsilon \rightarrow 0} F(v_{\varepsilon}; O) : W^{1,p}(\Omega; \mathbb{R}^m) \ni v_{\varepsilon} \rightarrow u \text{ in } L^p(\Omega; \mathbb{R}^m) \right\}; \\ \mathcal{F}_0^{\mathfrak{D}}(u; O) &:= \inf \left\{ \liminf_{\varepsilon \rightarrow 0} F(v_{\varepsilon}; O) : W_0^{1,p}(O; \mathbb{R}^m) + u \ni v_{\varepsilon} \rightarrow u \text{ in } L^p(\Omega; \mathbb{R}^m) \right\}; \\ m(u; O) &:= \inf \left\{ F(v; O) : v \in u + W_0^{1,p}(O; \mathbb{R}^m) \right\}. \end{aligned}$$

The following abstract relaxation result is a direct consequence of Remarks 2.3 (iii) and Theorem 2.2.

Proposition 2.2. *Let $O \in \mathcal{O}(\Omega)$ and let $u \in M_{\mathcal{F}}(O)$. If (H₂) and (H₃) hold then*

$$(2.4) \quad \mathcal{F}_0(u; O) = \mathcal{F}_0^{\mathfrak{D}}(u; O) = \int \lim_{\rho \rightarrow 0} \frac{m(u_x; Q_{\rho}(x))}{\rho^d} dx.$$

Remark 2.5. In particular (2.4) holds for all $u \in \text{dom}F(\cdot; O)$ since it is easy to see that $\text{dom}F(\cdot; O) \subset M_{\mathcal{F}}(O)$.

2.5. Remarks on the limit integrand. We assume that the assumptions of Theorem 2.2 hold. We give descriptions of the limit integrand L_0 by considering some particular cases.

(i) If we define $\tilde{L}_0 : \Omega \times \mathbb{R}^d \times \mathbb{M}^{m \times d} \rightarrow [0, \infty]$ by

$$(2.5) \quad \tilde{L}_0(x, v, \xi) := \overline{\lim}_{\rho \rightarrow 0} \frac{m_+(v + \xi(\cdot - x); \mathbb{Q}_\rho(x))}{\rho^d},$$

and for each $u \in M_{\mathcal{F}}(O)$

$$(2.6) \quad L_0(x, u(x), \nabla u(x)) := \lim_{\rho \rightarrow 0} \frac{m_+(u_x; \mathbb{Q}_\rho(x))}{\rho^d}$$

then the formula (2.2) becomes

$$\mathcal{F}_0(u; O) = \int_O \tilde{L}_0(x, u(x), \nabla u(x)) dx.$$

Indeed, we have for every $x \in O$

$$(2.7) \quad L_0(x, u(x), \nabla u(x)) = \tilde{L}_0(x, u(x), \nabla u(x)).$$

In fact, we do not know whether the integrand \tilde{L}_0 is Borel measurable. Because of the equality (2.7), the function $O \ni x \mapsto \tilde{L}_0(x, u(x), \nabla u(x))$ is measurable for all $u \in M_{\mathcal{F}}(O)$.

(ii) Assume that $\{F_\varepsilon\}_\varepsilon = \mathcal{F}$ is given under integral form, i.e., $F_\varepsilon : W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \rightarrow [0, \infty]$ is defined by

$$F_\varepsilon(u; O) := \int_O L_\varepsilon(x, u(x), \nabla u(x)) dx$$

where $L_\varepsilon : \Omega \times \mathbb{R}^d \times \mathbb{M}^{m \times d} \rightarrow [0, \infty]$ is Borel measurable for all $\varepsilon \in]0, 1]$. Then

$$(2.8) \quad \tilde{L}_0(x, v, \xi) = \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \inf \left\{ \int_{\mathbb{Q}_\rho(x)} L_\varepsilon(y, v + \xi(y - x) + \varphi(y), \xi + \nabla \varphi(y)) dy : \varphi \in W_0^{1,p}(\mathbb{Q}_\rho(x); \mathbb{R}^m) \right\}.$$

If, moreover, we assume that L_ε does not depend of the variable v , i.e., $L_\varepsilon : \Omega \times \mathbb{M}^{m \times d} \rightarrow [0, \infty]$ then (2.7) becomes

$$L_0(x, \nabla u(x)) = \tilde{L}_0(x, \nabla u(x))$$

for all $x \in O$ and all $u \in M_{\mathcal{F}}(O)$. Since the affine functions belong to $M_{\mathcal{F}}(O)$ we deduce that for every $x \in O$ and every $\xi \in \mathbb{M}^{m \times d}$

$$L_0(x, \xi) = \tilde{L}_0(x, \xi).$$

(iii) Now, we consider the case where $\{F_\varepsilon = F\}_\varepsilon = \mathcal{F}$ is constant with respect to ε and $F : W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \rightarrow [0, \infty]$ is defined by

$$F(u; O) := \int_O L(x, u(x), \nabla u(x)) dx$$

where $L : \Omega \times \mathbb{R}^d \times \mathbb{M}^{m \times d} \rightarrow [0, \infty]$ is Borel measurable. If we define for every $(x, v, \xi) \in \Omega \times \mathbb{R}^d \times \mathbb{M}^{m \times d}$

$$(2.9) \quad \tilde{L}_0(x, v, \xi) = \overline{\lim}_{\rho \rightarrow 0} \inf \left\{ \int_{\mathbb{Q}_\rho(x)} L(y, v + \xi(y - x) + \varphi(y), \xi + \nabla \varphi(y)) dy : \varphi \in W_0^{1,p}(\mathbb{Q}_\rho(x); \mathbb{R}^m) \right\}.$$

then for every $u \in M_{\mathcal{F}}(O)$ and every $x \in O$

$$L_0(x, u(x), \nabla u(x)) = \tilde{L}_0(x, u(x), \nabla u(x)).$$

We will show in Proposition 5.1 that when L is Carathéodory with p -growth and p -coercivity we recover the classical quasiconvex envelope [Dac08, Theorem 9.8, p. 432].

3. INTEGRAL REPRESENTATION OF VITALI ENVELOPE AND DERIVATION OF SET FUNCTIONS

3.1. Integral representation of Vitali envelope of set functions. For a given open set $O \subset \Omega$ we denote by $\mathcal{Q}_o(O)$ the set of all open cube of O . We denote by $\mathcal{Q}_c(O)$ the set of all closed cube of O .

Let $G : \mathcal{Q}_o(\Omega) \rightarrow]-\infty, \infty]$ be a set function. We define the *Vitali envelope* of G with respect to λ

$$\mathcal{O}(\Omega) \ni O \mapsto V_G(O) := \sup_{\varepsilon > 0} \inf \left\{ \sum_{i \in I} G(Q_i) : \{\overline{Q}_i\}_{i \in I} \in \mathcal{V}^\varepsilon(O) \right\}$$

where for any $\varepsilon > 0$

$$\mathcal{V}^\varepsilon(O) := \left\{ \{\overline{Q}_i\}_{i \in I} \subset \mathcal{Q}_c(\Omega) : I \text{ is countable, } \lambda\left(O \setminus \bigcup_{i \in I} Q_i\right) = 0, \overline{Q}_i \subset O \right. \\ \left. \text{diam}(Q_i) \in]0, \varepsilon[\text{ and } \overline{Q}_i \cap \overline{Q}_j = \emptyset \text{ for all } i \neq j \right\}.$$

Remark 3.1. If G is the trace on $\mathcal{Q}_o(\Omega)$ of a positive Borel measure ν on Ω which is absolutely continuous with respect to λ then $V_G(O) = \nu(O)$ for all $O \in \mathcal{O}(\Omega)$.

Let $G : \mathcal{Q}_o(\Omega) \rightarrow]-\infty, \infty]$ be a set function. Define the upper and the lower derivatives at $x \in \Omega$ of G with respect to λ as follows

$$\underline{D}_\lambda G(x) := \liminf_{\rho \rightarrow 0} \left\{ \frac{G(Q)}{\lambda(Q)} : x \in Q \in \mathcal{Q}_o(\Omega), \text{diam}(Q) \leq \rho \right\}; \\ \overline{D}_\lambda G(x) := \limsup_{\rho \rightarrow 0} \left\{ \frac{G(Q)}{\lambda(Q)} : x \in Q \in \mathcal{Q}_o(\Omega), \text{diam}(Q) \leq \rho \right\}.$$

We say that G is λ -differentiable in O if for λ -almost all $x \in O$ it holds

$$-\infty < \underline{D}_\lambda G(x) = \overline{D}_\lambda G(x) < \infty.$$

Remark 3.2. For every $O \in \mathcal{O}(\Omega)$ and every $x \in O$ we have

$$\underline{D}_\lambda G(x) := \liminf_{\rho \rightarrow 0} \left\{ \frac{G(Q)}{\lambda(Q)} : x \in Q \in \mathcal{Q}_o(O), \text{diam}(Q) \leq \rho \right\}; \\ \overline{D}_\lambda G(x) := \limsup_{\rho \rightarrow 0} \left\{ \frac{G(Q)}{\lambda(Q)} : x \in Q \in \mathcal{Q}_o(O), \text{diam}(Q) \leq \rho \right\}.$$

The proof of the following classical result can be found in Appendix.

Lemma 3.1. *The functions $\underline{D}_\lambda G(\cdot)$ and $\overline{D}_\lambda G(\cdot)$ are λ -measurable.*

Remark 3.3. When $G = \nu$ is a Borel finite measure absolutely continuous with respect to λ then ν is λ -differentiable in O and

$$D_\lambda \nu(x) = \lim_{\rho \rightarrow 0} \frac{\nu(Q_\rho(x))}{\rho^d} \quad \text{a.e. in } O.$$

The following result establishes an integral representation for the Vitali envelope of nonnegative set functions.

Proposition 3.1. *Let $H : \mathcal{Q}_o(\Omega) \rightarrow [0, \infty]$ be a set function. For every $O \in \mathcal{O}(\Omega)$ we have*

$$V_H(O) = \int_O \underline{D}_\lambda H(y) dy.$$

The following lemma was inspired by reading [Bon82].

Lemma 3.2. *Let $G : \mathcal{Q}_o(\Omega) \rightarrow]-\infty, \infty]$ be a set function and $O \in \mathcal{O}(\Omega)$.*

- (a) *If $\underline{D}_\lambda G(x) \leq 0$ λ -a.e. in O then $V_G(O) \leq 0$.*
- (b) *If $\underline{D}_\lambda G(x) \geq 0$ λ -a.e. in O then $V_G(O) \geq 0$.*
- (c) *If $\underline{D}_\lambda G(x) = 0$ λ -a.e. in O then $V_G(O) = 0$.*

Proof. The assertion (c) is a consequence of (a) and (b).

Proof of (a). It is enough to show that for every $\varepsilon > 0$ if

$$(3.1) \quad \underline{D}_\lambda G(x) < \varepsilon \quad \lambda\text{-a.e. in } O$$

then

$$\inf \left\{ \sum_{i \in I} G(Q_i) : \{\overline{Q}_i\}_{i \in I} \in \mathcal{V}_\varepsilon(O) \right\} < \varepsilon \lambda(O).$$

Fix $\varepsilon > 0$. Let $N \subset O$ with $\lambda(N) = 0$ be such that $O \setminus N = [\underline{D}_\lambda G(\cdot) < \varepsilon]$. Using Lemma 6.1 (i) with $h = \eta = \varepsilon$ and $S_h = O \setminus N$, there exists a countable pairwise disjointed family $\{Q_i\}_{i \in I} \subset \mathcal{Q}_o(O)$ such that

$$(3.2) \quad \lambda \left((O \setminus N) \setminus \bigcup_{i \in I} Q_i \right) = 0, \quad \forall i \in I \quad G(Q_i) < \varepsilon \lambda(Q_i) \text{ and } \text{diam}(Q_i) < \varepsilon.$$

From (3.2) we have $\lambda(O \setminus \bigcup_{i \in I} \overline{Q}_i) = 0$ since $\lambda(N) = 0$. Consequently, we have $\sum_{i \in I} \lambda(\overline{Q}_i) = \sum_{i \in I} \lambda(Q_i) = \lambda(O)$. Summing over $i \in I$ the first inequality (3.2) we obtain

$$\inf \left\{ \sum_{i \in I} G(Q_i) : \{\overline{Q}_i\}_{i \in I} \in \mathcal{V}_\varepsilon(O) \right\} < \varepsilon \sum_{i \in I} \lambda(Q_i) = \varepsilon \lambda(O).$$

Proof of (b). Let $\{\varepsilon_n\}_{n \in \mathbb{N}} \subset]0, 1[$ be such that $\lim_{n \rightarrow \infty} \varepsilon_n = 0$. Let $n \in \mathbb{N}$. There exists $\{Q_i^n\}_{i \in I_n} \in \mathcal{V}_{\varepsilon_n}(O)$ such that

$$(3.3) \quad \inf \left\{ \sum_{i \in I} G(Q_i) : \{Q_i\} \in \mathcal{V}_{\varepsilon_n}(O) \right\} + \frac{1}{n} \geq \sum_{i \in I_n} \frac{G(Q_i^n)}{\lambda(Q_i^n)} \lambda(Q_i^n).$$

Fix $x \in O$ be such that

$$\underline{D}_\lambda G(x) \geq 0 \quad \text{and} \quad x \notin \bigcup_{n \in \mathbb{N}} \left(O \setminus \bigcup_{i \in I_n} Q_i^n \right).$$

There exists $i_x^n \in I_n$ such that $x \in Q_{i_x^n}^n$. From (3.3) it follows that

$$(3.4) \quad \begin{aligned} & \inf \left\{ \sum_{i \in I} G(Q_i) : \{Q_i\} \in \mathcal{V}_{\varepsilon_n}(O) \right\} + \frac{1}{n} \\ & \geq \frac{G(Q_{i_x^n}^n)}{\lambda(Q_{i_x^n}^n)} \lambda(Q_{i_x^n}^n) \\ & \geq \inf \left\{ \frac{G(Q)}{\lambda(Q)} : x \in Q \in \mathcal{Q}_o(O), \text{diam}(Q) \leq \varepsilon_n \right\} \lambda(Q_{i_x^n}^n). \end{aligned}$$

Passing to the limit $n \rightarrow \infty$ in (3.4) we obtain

$$V_G(O) \geq \underline{D}_\lambda G(x) \lim_{n \rightarrow \infty} \lambda(Q_{i_x^n}^n) \geq 0.$$

The proof is complete. ■

Corollary 3.1. *If $G = H - \nu$ where $H : \mathcal{Q}_o(\Omega) \rightarrow]-\infty, \infty]$ is a set function, ν is a finite Borel measure on $O \in \mathcal{O}(\Omega)$ absolutely continuous with respect to $\lambda|_O$, and if*

$$\underline{D}_\lambda G(x) = 0 \quad \lambda\text{-a.e. in } O$$

then

$$V_H(O) = \nu(O).$$

Proof. Use Lemma 3.2(c) and remark that $V_G(O) = V_H(O) - \nu(O)$. ■

Proof of Proposition 3.1. Assume that $\underline{D}_\lambda H \in L^1(O)$, i.e.,

$$\int_O \underline{D}_\lambda H(y) dy < \infty.$$

Define $G : \mathcal{Q}_o(O) \rightarrow]-\infty, \infty]$ by

$$G(Q) := H(Q) - \int_Q \underline{D}_\lambda H(y) dy.$$

If we show that $\underline{D}_\lambda G = 0$ a.e. in O then by Corollary 3.1 we can conclude that

$$V_H(O) = \nu(O) \text{ with } \nu := \underline{D}_\lambda H(\cdot) \lambda|_O.$$

Fix $x \in O$ such that

$$(3.5) \quad \underline{D}_\lambda H(x) = \lim_{\rho \rightarrow 0} \int_{Q_\rho(x)} \underline{D}_\lambda H(y) dy < \infty;$$

$$(3.6) \quad \underline{D}_\lambda \nu(x) = \underline{D}_\lambda H(x) < \infty.$$

We set $\mathfrak{B}_{x,\rho} := \{Q : x \in Q \in \mathcal{Q}_o(O) \text{ and } \text{diam}(Q) \leq \rho\}$ for all $\rho > 0$. On one hand, we have for every $\rho > 0$ and every $Q \in \mathfrak{B}_{x,\rho}$

$$\begin{aligned} \frac{G(Q)}{\lambda(Q)} &\leq \frac{H(Q)}{\lambda(Q)} + \sup \left\{ - \int_Q \underline{D}_\lambda H(y) dy : Q \in \mathfrak{B}_{x,\rho} \right\} \\ &= \frac{H(Q)}{\lambda(Q)} - \inf \left\{ \int_Q \underline{D}_\lambda H(y) dy : Q \in \mathfrak{B}_{x,\rho} \right\}. \end{aligned}$$

Taking the infimum over every $Q \in \mathfrak{B}_{x,\rho}$ we obtain

$$\begin{aligned} &\inf \left\{ \frac{G(Q)}{\lambda(Q)} : Q \in \mathfrak{B}_{x,\rho} \right\} \\ &\leq \inf \left\{ \frac{H(Q)}{\lambda(Q)} : Q \in \mathfrak{B}_{x,\rho} \right\} - \inf \left\{ \int_Q \underline{D}_\lambda H(y) dy : Q \in \mathfrak{B}_{x,\rho} \right\}. \end{aligned}$$

Letting $\rho \rightarrow 0$ and using (3.5) and (3.6), we have

$$(3.7) \quad \underline{D}_\lambda G(x) \leq \underline{D}_\lambda H(x) - \underline{D}_\lambda H(x) = 0.$$

On the other hand, we have for every $\rho > 0$ and every $Q \in \mathfrak{B}_{x,\rho}$

$$\begin{aligned} \frac{G(Q)}{\lambda(Q)} &\geq \frac{H(Q)}{\lambda(Q)} + \inf \left\{ - \int_Q \underline{D}_\lambda H(y) dy : Q \in \mathfrak{B}_{x,\rho} \right\} \\ &= \frac{H(Q)}{\lambda(Q)} - \sup \left\{ \int_Q \underline{D}_\lambda H(y) dy : Q \in \mathfrak{B}_{x,\rho} \right\}. \end{aligned}$$

Taking the infimum over every $Q \in \mathfrak{B}_{x,\rho}$ we obtain

$$\begin{aligned} &\inf \left\{ \frac{G(Q)}{\lambda(Q)} : Q \in \mathfrak{B}_{x,\rho} \right\} \\ &\leq \inf \left\{ \frac{H(Q)}{\lambda(Q)} : Q \in \mathfrak{B}_{x,\rho} \right\} - \sup \left\{ \int_Q \underline{D}_\lambda H(y) dy : Q \in \mathfrak{B}_{x,\rho} \right\}. \end{aligned}$$

Letting $\rho \rightarrow 0$ and using (3.5) and (3.6), we have

$$(3.8) \quad \underline{D}_\lambda G(x) \geq \underline{D}_\lambda H(x) - \underline{D}_\lambda H(x) = 0.$$

Taking account of (3.7) and (3.8), we finally obtain that $\underline{D}_\lambda G(x) = 0$.

Now, we do not assume that $\underline{D}_\lambda H \in L^1(O)$, in this case the following inequality is always true

$$V_H(O) \leq \int_O \underline{D}_\lambda H(y) dy.$$

It remains to prove the opposite inequality. For every $n \in \mathbb{N}$ we set $H_n : \mathcal{Q}_0(\Omega) \rightarrow [0, \infty[$ defined by

$$H_n(Q) := \begin{cases} H(Q) & \text{if } H(Q) \leq n\lambda(Q) \\ n\lambda(Q) & \text{if } H(Q) > n\lambda(Q). \end{cases}$$

It is easy to see that

$$(3.9) \quad \begin{aligned} \forall Q \in \mathcal{Q}_0(\Omega) \quad H_0(Q) &\leq H_1(Q) \leq \dots \leq H_n(Q) \leq \dots \leq \sup_{n \in \mathbb{N}} H_n(Q) \leq H(Q); \\ \forall n \in \mathbb{N} \quad \underline{D}_\lambda H_n &\leq n. \end{aligned}$$

So $\{\underline{D}_\lambda H_n\}_{n \in \mathbb{N}} \subset L^1(O)$, we apply the first part of the proof to have

$$\forall n \in \mathbb{N} \quad V_{H_n}(O) = \int_O \underline{D}_\lambda H_n(y) dy \leq V_H(O).$$

Using (3.9) and monotone convergence theorem we have

$$(3.10) \quad \sup_{n \in \mathbb{N}} V_{H_n}(O) = \int_O \sup_{n \in \mathbb{N}} \underline{D}_\lambda H_n(y) dy \leq V_H(O).$$

Fix $n \in \mathbb{N}$ and $x \in [\underline{D}_\lambda H \leq n]$. Then for every $\rho > 0$ we have

$$(3.11) \quad \inf_{Q \in \mathfrak{B}_{x,\rho}} \frac{H(Q)}{\lambda(Q)} \leq n.$$

For each $n \in \mathbb{N}$ and each $\rho > 0$ we set $A_n := \{Q \in \mathfrak{B}_{x,\rho} : H(Q) \leq n\lambda(Q)\}$ and $B_n := \mathfrak{B}_{x,\rho} \setminus A_n$. Then

$$\begin{aligned} \underline{D}_\lambda H_n(x) &= \sup_{\rho > 0} \inf_{Q \in \mathfrak{B}_{x,\rho}} \frac{H_n(Q)}{\lambda(Q)} \\ &= \sup_{\rho > 0} \min \left\{ \inf_{Q \in A_n} \frac{H_n(Q)}{\lambda(Q)}, \inf_{Q \in B_n} \frac{H_n(Q)}{\lambda(Q)} \right\} \\ &= \sup_{\rho > 0} \min \left\{ \inf_{Q \in A_n} \frac{H(Q)}{\lambda(Q)}, n \right\} \\ &\geq \sup_{\rho > 0} \min \left\{ \inf_{Q \in \mathfrak{B}_{x,\rho}} \frac{H(Q)}{\lambda(Q)}, n \right\}. \end{aligned}$$

Using (3.11) we find

$$\underline{D}_\lambda H_n(x) \geq \sup_{\rho > 0} \inf_{Q \in \mathfrak{B}_{x,\rho}} \frac{H(Q)}{\lambda(Q)} = \underline{D}_\lambda H(x).$$

It follows that $\sup_{n \in \mathbb{N}} \underline{D}_\lambda H_n(x) = \underline{D}_\lambda H(x)$ for all $x \in O$ and thus (3.10) becomes

$$\int_O \underline{D}_\lambda H(y) dy \leq V_H(O).$$

The proof is complete. \blacksquare

4. PROOF OF MAIN RESULTS

4.1. Proof of Lemma 2.1. Fix $O \in \mathcal{O}(\Omega)$ and $u \in M_{\mathcal{F}}(O)$. By Theorem 2.1 (ii) there exists $\{u_{\varepsilon_n}\}_{\varepsilon_n}$ with $\sup_n F_{\varepsilon_n}(u_{\varepsilon_n}; \Omega) < \infty$ such that $u_{\varepsilon_n} \rightarrow u$ in $L^p(\Omega; \mathbb{R}^m)$ as $n \rightarrow \infty$ and

$$\infty > \mathcal{F}_-^{\mathfrak{D}}(u; O) \geq \mathcal{F}_-(u; O) \geq \int_O \lim_{\rho \rightarrow 0} \lim_{n \rightarrow \infty} \frac{F_{\varepsilon_n}(u_{\varepsilon_n}; Q_\rho(x))}{\rho^d} dx.$$

since Remark 2.2. It follows that for almost all $x \in O$

$$(4.1) \quad \liminf_{\rho \rightarrow 0} \frac{\mathcal{F}_-(u; Q_\rho(x))}{\rho^d} \geq \lim_{\rho \rightarrow 0} \lim_{n \rightarrow \infty} \frac{F_{\varepsilon_n}(u_{\varepsilon_n}; Q_\rho(x))}{\rho^d}.$$

Using the local inequalities (H₁), (H₂), (H₃), (4.1) and Theorem 2.1 (i) we have for almost all $x \in O$

$$\begin{aligned}
\liminf_{\rho \rightarrow 0} \frac{\mathcal{F}_+(u; \mathbb{Q}_\rho(x))}{\rho^d} &\leq \liminf_{\rho \rightarrow 0} \frac{\mathcal{F}_+^\mathfrak{D}(u; \mathbb{Q}_\rho(x))}{\rho^d} \\
&\leq \underline{D}_\lambda m_+(u; \cdot)(x) \\
&\leq \liminf_{\rho \rightarrow 0} \frac{m_+(u; \mathbb{Q}_\rho(x))}{\rho^d} \\
&\leq \liminf_{\rho \rightarrow 0} \frac{m_+(u_x; \mathbb{Q}_\rho(x))}{\rho^d} \\
&\leq \liminf_{\rho \rightarrow 0} \frac{m_+(u_x; \mathbb{Q}_\rho(x))}{\rho^d} \\
&\leq \liminf_{\rho \rightarrow 0} \frac{m_-(u_x; \mathbb{Q}_\rho(x))}{\rho^d} \\
&\leq \lim_{\rho \rightarrow 0} \lim_{n \rightarrow \infty} \frac{F_{\varepsilon_n}(u_{\varepsilon_n}; \mathbb{Q}_\rho(x))}{\rho^d} \\
(4.2) \quad &\leq \liminf_{\rho \rightarrow 0} \frac{\mathcal{F}_-(u; \mathbb{Q}_\rho(x))}{\rho^d}.
\end{aligned}$$

From the last inequality (4.2) we have the following inequalities

$$\begin{aligned}
\liminf_{\rho \rightarrow 0} \frac{\mathcal{F}_-(u; \mathbb{Q}_\rho(x))}{\rho^d} &\leq \liminf_{\rho \rightarrow 0} \frac{\mathcal{F}_+(u; \mathbb{Q}_\rho(x))}{\rho^d} \leq \liminf_{\rho \rightarrow 0} \frac{\mathcal{F}_+(u; \mathbb{Q}_\rho(x))}{\rho^d} \\
&\text{and} \\
\liminf_{\rho \rightarrow 0} \frac{\mathcal{F}_-(u; \mathbb{Q}_\rho(x))}{\rho^d} &\leq \liminf_{\rho \rightarrow 0} \frac{\mathcal{F}_-(u; \mathbb{Q}_\rho(x))}{\rho^d} \leq \liminf_{\rho \rightarrow 0} \frac{\mathcal{F}_+(u; \mathbb{Q}_\rho(x))}{\rho^d} \\
&\text{and} \\
\liminf_{\rho \rightarrow 0} \frac{\mathcal{F}_-(u; \mathbb{Q}_\rho(x))}{\rho^d} &\leq \liminf_{\rho \rightarrow 0} \frac{\mathcal{F}_+^\mathfrak{D}(u; \mathbb{Q}_\rho(x))}{\rho^d} \leq \liminf_{\rho \rightarrow 0} \frac{\mathcal{F}_+^\mathfrak{D}(u; \mathbb{Q}_\rho(x))}{\rho^d} \\
&\text{and} \\
\liminf_{\rho \rightarrow 0} \frac{\mathcal{F}_-(u; \mathbb{Q}_\rho(x))}{\rho^d} &\leq \liminf_{\rho \rightarrow 0} \frac{\mathcal{F}_-^\mathfrak{D}(u; \mathbb{Q}_\rho(x))}{\rho^d} \leq \liminf_{\rho \rightarrow 0} \frac{\mathcal{F}_-^\mathfrak{D}(u; \mathbb{Q}_\rho(x))}{\rho^d} \\
&\leq \liminf_{\rho \rightarrow 0} \frac{\mathcal{F}_+^\mathfrak{D}(u; \mathbb{Q}_\rho(x))}{\rho^d}
\end{aligned}$$

for all $x \in O$. It follows that for almost all $x \in O$

$$\begin{aligned}
\lim_{\rho \rightarrow 0} \frac{\mathcal{F}_-(u; \mathbb{Q}_\rho(x))}{\rho^d} &= \lim_{\rho \rightarrow 0} \frac{\mathcal{F}_+(u; \mathbb{Q}_\rho(x))}{\rho^d} \\
&= \lim_{\rho \rightarrow 0} \frac{\mathcal{F}_+^\mathfrak{D}(u; \mathbb{Q}_\rho(x))}{\rho^d} = \lim_{\rho \rightarrow 0} \frac{\mathcal{F}_-^\mathfrak{D}(u; \mathbb{Q}_\rho(x))}{\rho^d} \\
&= \underline{D}_\lambda m_+(u; \cdot)(x) = \liminf_{\rho \rightarrow 0} \frac{m_+(u; \mathbb{Q}_\rho(x))}{\rho^d} \\
&= \lim_{\rho \rightarrow 0} \frac{m_+(u_x; \mathbb{Q}_\rho(x))}{\rho^d} = \liminf_{\rho \rightarrow 0} \frac{m_-(u_x; \mathbb{Q}_\rho(x))}{\rho^d}.
\end{aligned}$$

So, the proof is complete since $O \ni x \mapsto \underline{D}_\lambda m_+(u; \cdot)(x)$ is measurable by Lemma 3.1. \blacksquare

4.2. Proof of Theorem 2.1.

Proof of Theorem 2.1 (ii). Let $(u, O) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega)$ be such that $\mathcal{F}_-(u; O) < \infty$. There exists a sequence $\{u_{\varepsilon_n}\}_n \subset W^{1,p}(\Omega; \mathbb{R}^m)$ such that

$$(4.3) \quad u_{\varepsilon_n} \rightarrow u \text{ in } L^p(\Omega; \mathbb{R}^m), \quad \lim_{n \rightarrow \infty} F_{\varepsilon_n}(u_{\varepsilon_n}; O) = \mathcal{F}_-(u; O) \quad \text{and} \quad \sup_n F_{\varepsilon_n}(u_{\varepsilon_n}; O) < \infty.$$

By (C₂), for each $\varepsilon > 0$ we consider the Borel measure ν_ε whose the trace on $\mathcal{O}(\Omega)$ is $F_\varepsilon(u_\varepsilon; \cdot)$. From the last inequality of (4.3) we can rewrite that the sequence of Borel measures $\{\mu_n := \nu_{\varepsilon_n}|_O\}_n$ satisfies $\sup_n \mu_n(O) < \infty$. So, there exists a Borel measure μ on O such that (up to a subsequence) $\mu_n \xrightarrow{*} \mu$.

By Lebesgue decomposition theorem, we have $\mu = \mu_a + \mu_s$ where μ_a and μ_s are nonnegative Borel measures such that $\mu_a \ll \lambda|_O$ and $\mu_s \perp \lambda|_O$, and from Radon-Nikodym theorem we deduce that there exists $f \in L^1(O; \mathbb{R}^+)$, given by

$$f(x) = \lim_{\rho \rightarrow 0} \frac{\mu_a(Q_\rho(x))}{\rho^d} = \lim_{\rho \rightarrow 0} \frac{\mu(Q_\rho(x))}{\rho^d} \quad \text{a.e. in } O$$

with $Q_\rho(x) := x + \rho Y$, such that

$$\mu_a(A) = \int_A f(x) dx \quad \text{for all measurable sets } A \subset O.$$

By Alexandrov theorem we see that

$$\begin{aligned} \mathcal{F}_-(u; O) &= \lim_{n \rightarrow \infty} F_{\varepsilon_n}(u_{\varepsilon_n}; O) \\ &= \lim_{n \rightarrow \infty} \mu_n(O) \geq \mu(O) = \mu_a(O) + \mu_s(O) \geq \mu_a(O) = \int_O f(x) dx, \end{aligned}$$

and

$$f(x) = \lim_{\rho \rightarrow 0} \lim_{n \rightarrow \infty} \frac{\mu_n(Q_\rho(x))}{\rho^d} = \lim_{\rho \rightarrow 0} \lim_{n \rightarrow \infty} \frac{F_{\varepsilon_n}(u_{\varepsilon_n}; Q_\rho(x))}{\rho^d} \quad \text{a.e. in } O. \quad \blacksquare$$

Proof of Theorem 2.1 (i). For each $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ we denote by $V_+(u; \cdot) : \mathcal{O}(\Omega) \rightarrow [0, \infty]$ the Vitali envelope of $m_+(u; \cdot)$, i.e.,

$$V_+(u; O) := V_{m_+(u; \cdot)}(O).$$

The proof consists to show that for every $O \in \mathcal{O}(\Omega)$ and every $u \in M_{\mathcal{F}}(O)$ the following inequality holds

$$(4.4) \quad \mathcal{F}_+^{\mathfrak{D}}(u; O) \leq V_+(u; O).$$

Indeed, using Proposition 3.1 we obtain

$$\mathcal{F}_+(u; O) \leq \mathcal{F}_+^{\mathfrak{D}}(u; O) \leq V_+(u; O) = \int_O \underline{D}_\lambda m_+(u; \cdot)(x) dx.$$

Let us prove (4.4) now. Fix $O \in \mathcal{O}(\Omega)$ and $u \in M_{\mathcal{F}}(O)$. Note that by Remarks 3.1 we have for some $\mu_u \in \mathfrak{A}_\lambda(O)$

$$(4.5) \quad V_+(u; O) \leq \mu_u(O) < \infty.$$

Fix $\varepsilon \in]0, 1[$. Choose $\{\overline{Q}_i\}_{i \in I} \in \mathcal{V}_\varepsilon(O)$ such that

$$(4.6) \quad \sum_{i \in I} m_+(u; Q_i) \leq V_+^\varepsilon(u; O) + \frac{\varepsilon}{2} \leq V_+(u; O) + \frac{\varepsilon}{2}.$$

Fix $\delta \in]0, 1[$. Given any $i \in I$, by definition of $m_\delta(u; Q_i)$, there exists $v_i \in u + W_0^{1,p}(Q_i; \mathbb{R}^m)$ such that

$$(4.7) \quad F_\delta(v_i; Q_i) \leq m_\delta(u; Q_i) + \frac{\delta \lambda(Q_i)}{2 \lambda(O)}.$$

Define $u_{\delta, \varepsilon} \in u + W_0^{1,p}(O; \mathbb{R}^m)$ by

$$u_{\delta, \varepsilon} := \sum_{i \in I} v_i \mathbf{1}_{Q_i} + u \mathbf{1}_{\Omega \setminus \bigcup_{i \in I} Q_i}.$$

Using (C₂) and (C₃) we have from (4.7)

$$\begin{aligned} F_\delta(u_{\delta, \varepsilon}; O) &= \sum_{i \in I} F_\delta(v_i; Q_i) + F_\delta\left(u; O \setminus \bigcup_{i \in I} Q_i\right) \\ &= \sum_{i \in I} F_\delta(v_i; Q_i) \\ &\leq \sum_{i \in I} m_\delta(u; Q_i) + \frac{\delta}{2}. \end{aligned}$$

Since $u \in M_{\mathcal{F}}(O)$ there exists $\mu_u \in \mathfrak{A}_{\lambda}(O)$ such that $\sup_{\delta \in]0,1]} m_{\delta}(u; U) \leq \mu_u(U)$ for all open set $U \subset O$. For every $\eta > 0$ there exists a finite set $I_{\eta} \subset I$ such that $\mu_u(O \setminus \cup_{i \in I_{\eta}} Q_i) \leq \eta$. It follows that $\sum_{i \in I \setminus I_{\eta}} m_{\delta}(u; Q_i) \leq \eta$. Hence, for any $\eta > 0$

$$(4.8) \quad \begin{aligned} \overline{\lim}_{\delta \rightarrow 0} \sum_{i \in I} m_{\delta}(u; Q_i) &\leq \overline{\lim}_{\delta \rightarrow 0} \sum_{i \in I_{\eta}} m_{\delta}(u; Q_i) + \overline{\lim}_{\delta \rightarrow 0} \sum_{i \in I \setminus I_{\eta}} m_{\delta}(u; Q_i) \\ &\leq \sum_{i \in I} m_{+}(u; Q_i) + \eta. \end{aligned}$$

Therefore collecting (4.6), (4.8), and passing to the limit $\varepsilon \rightarrow 0$, we have

$$(4.9) \quad \overline{\lim}_{\varepsilon \rightarrow 0} \overline{\lim}_{\delta \rightarrow 0} F_{\delta}(u_{\delta, \varepsilon}; O) \leq V_{+}(u; O).$$

From the p -coercivity (C₁), (4.9) and (4.5), we deduce

$$(4.10) \quad \overline{\lim}_{\varepsilon \rightarrow 0} \overline{\lim}_{\delta \rightarrow 0} \int_O |\nabla u_{\delta, \varepsilon}|^p dx < \infty.$$

By Poincaré inequality there exists $K > 0$ depending only on p and d such that for each $v_i \in u + W_0^{1,p}(Q_i; \mathbb{R}^m)$

$$\int_{Q_i} |v_i - u|^p dx \leq K \varepsilon^p \int_{Q_i} |\nabla v_i - \nabla u|^p dx$$

since $\text{diam}(Q_i) < \varepsilon$. Summing over $i \in I$ we obtain

$$\int_O |u_{\delta, \varepsilon} - u|^p dx \leq 2^{p-1} K \varepsilon^p \left(\int_O |\nabla u_{\delta, \varepsilon}|^p dx + \int_O |\nabla u|^p dx \right)$$

which shows, by using (4.10), that

$$(4.11) \quad \overline{\lim}_{\varepsilon \rightarrow 0} \overline{\lim}_{\delta \rightarrow 0} \int_{\Omega} |u_{\delta, \varepsilon} - u|^p dx = 0.$$

A simultaneous diagonalization of (4.9) and (4.11) gives a sequence $\{u_{\delta} := u_{\delta, \varepsilon(\delta)}\}_{\delta} \subset u + W^{1,p}(O; \mathbb{R}^m)$ such that $u_{\delta} \rightarrow u$ in $L^p(\Omega; \mathbb{R}^m)$ and

$$\mathcal{F}_{+}^{\mathfrak{D}}(u; O) \leq \overline{\lim}_{\delta \rightarrow 0} F_{\delta}(u_{\delta}; O) \leq V_{+}(u; O)$$

by the definition of $\mathcal{F}_{+}^{\mathfrak{D}}(u; O)$. The proof is complete. \blacksquare

5. APPLICATIONS

5.1. General $\Gamma(L^p)$ -convergence result in the p -growth case. For each $\varepsilon \in]0,1]$ we consider a family of functionals $\mathcal{F} := \{F_{\varepsilon}\}_{\varepsilon \in]0,1]}$, $F_{\varepsilon} : W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \rightarrow [0, \infty]$.

Consider the following condition:

(P₁) there exist $\beta > 0$ and ν a nonnegative finite Borel measure on Ω absolutely continuous with respect to the Lebesgue measure such that for every $(V, u, \varepsilon) \in \mathcal{O}(\Omega) \times W^{1,p}(\Omega; \mathbb{R}^m) \times]0,1]$ we have

$$\frac{m_{\varepsilon}(u; V)}{|V|} \leq \beta \left(\frac{\nu(V)}{|V|} + \int_V |u|^p dx + \int_V |\nabla u|^p dx \right)$$

The following result can be seen as a nonconvex extension of Theorem IV of [DMM86a, p. 265]. Indeed, if for each $\varepsilon > 0$ we set $F_{\varepsilon} : W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \rightarrow [0, \infty]$ defined by

$$F_{\varepsilon}(u; O) := \int_O L_{\varepsilon}(x, u(x), \nabla u(x)) dx$$

where $L_{\varepsilon} : \Omega \times \mathbb{R}^d \times \mathbb{M}^{m \times d} \rightarrow [0, \infty[$ is a Borel measurable function with p -growth and p -coercivity, i.e.,

$$(5.1) \quad \exists \alpha > 0 \quad \exists \beta > 0 \quad \exists a \in L^1(\Omega) \quad \forall (x, v, \xi) \in \Omega \times \mathbb{R}^d \times \mathbb{M}^{m \times d} \quad \forall \varepsilon > 0$$

$$\alpha |\xi|^p \leq L_{\varepsilon}(x, v, \xi) \leq \beta (a(x) + |v|^p + |\xi|^p)$$

then (P₁) holds with $\nu = a(\cdot) \lambda$ and $\mathcal{F} = \{F_{\varepsilon}\}_{\varepsilon} \subset \mathcal{I}(p, \alpha)$.

Theorem 5.1. Assume that $\mathcal{F} \subset \mathcal{I}(p, \alpha)$. Let $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ and $O \in \mathcal{O}(\Omega)$. If (H_1) and (P_1) hold then the family $\mathcal{F}(\cdot; O)$ $\Gamma(L^p)$ -converges at u to

$$\mathcal{F}_0(u, O) := \int_O L_0(x, u(x), \nabla u(x)) dx$$

where $L_0(\cdot, u(\cdot), \nabla u(\cdot))$ is given by (2.6).

Proof. Since (P_1) we see that

$$M_{\mathcal{F}}(O) = W^{1,p}(\Omega; \mathbb{R}^m).$$

Fix $u \in W^{1,p}(\Omega; \mathbb{R}^m)$. Following Theorem 2.2 it is enough to show that (H_2) and (H_3) hold.

We begin by showing (H_3) . Fix $x \in O$ such that

$$(5.2) \quad \lim_{r \rightarrow 0} \int_{Q_r(x)} |u|^p dy = |u(x)|^p < \infty;$$

$$(5.3) \quad \lim_{r \rightarrow 0} \int_{Q_r(x)} |\nabla u|^p dy = |\nabla u(x)|^p < \infty;$$

$$(5.4) \quad \lim_{r \rightarrow 0} \frac{1}{r^p} \int_{Q_r(x)} |u_x - u|^p dy = 0;$$

$$(5.5) \quad \lim_{r \rightarrow 0} \frac{\nu(Q_r(x))}{r^d} = D_\lambda \nu(x) < \infty.$$

Fix $\varepsilon > 0$, $s \in]0, 1[$ and $\rho > 0$. Let $\phi \in W_0^{1,\infty}(Q_\rho(x); [0, 1])$ be a cut-off function between $\overline{Q_{s\rho}}(x)$ and $Q_\rho(x)$ (i.e., $\phi = 1$ on $\overline{Q_{s\rho}}(x)$ and $\phi = 0$ on $O \setminus Q_{s\rho}(x)$) such that

$$\|\nabla \phi\|_{L^\infty(Q_\rho(x))} \leq \frac{4}{\rho(1-s)}.$$

Let $v_\varepsilon \in u_x + W_0^{1,p}(Q_{s\rho}(x); \mathbb{R}^m)$ be such that

$$(5.6) \quad F_\varepsilon(v_\varepsilon; Q_{s\rho}(x)) \leq \varepsilon (s\rho)^d + m_\varepsilon(u_x; Q_{s\rho}(x)).$$

Set $w := \phi v_\varepsilon + (1 - \phi)u$, we have $w \in u + W_0^{1,p}(Q_\rho(x); \mathbb{R}^m)$ and

$$\nabla w := \begin{cases} \nabla v_\varepsilon & \text{in } Q_{s\rho}(x) \\ \phi \nabla u(x) + (1 - \phi) \nabla u + \nabla \phi \otimes (u_x - u) & \text{in } \Sigma_\rho(x) \end{cases}$$

where $\Sigma_\rho(x) := Q_\rho(x) \setminus \overline{Q_{s\rho}}(x)$. We have

$$\begin{aligned} m_\varepsilon(u; Q_\rho(x)) &= m_\varepsilon(w; Q_\rho(x)) \\ &= m_\varepsilon(w; Q_{s\rho}(x)) + m_\varepsilon(w; \Sigma_\rho(x)) \\ &\leq F_\varepsilon(v_\varepsilon; Q_{s\rho}(x)) + m_\varepsilon(w; \Sigma_\rho(x)) \\ &\leq \varepsilon (s\rho)^d + m_\varepsilon(u_x; Q_{s\rho}(x)) + m_\varepsilon(w; \Sigma_\rho(x)) \end{aligned}$$

since Lemma 6.2 and (5.6). It follows that

$$(5.7) \quad \frac{m_\varepsilon(u; Q_\rho(x))}{\rho^d} \leq \varepsilon s^d + s^d \frac{m_\varepsilon(u_x; Q_{s\rho}(x))}{(s\rho)^d} + \frac{m_\varepsilon(w; \Sigma_\rho(x))}{\rho^d}.$$

We claim that (H_3) is proved if

$$(5.8) \quad \overline{\lim}_{s \rightarrow 1} \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(w; \Sigma_\rho(x))}{\rho^d} = 0.$$

Indeed, passing to the limits $\varepsilon \rightarrow 0$, $\rho \rightarrow 0$, $s \rightarrow 1$ in (5.7) we have

$$(5.9) \quad \begin{aligned} \underline{\lim}_{\rho \rightarrow 0} \frac{m_+(u; Q_\rho(x))}{\rho^d} &\leq \underline{\lim}_{s \rightarrow 1^-} \underline{\lim}_{\rho \rightarrow 0} \frac{m_+(u_x; Q_{s\rho}(x))}{(s\rho)^d} \\ &\leq \underline{\lim}_{\rho \rightarrow 0} \frac{m_+(u_x; Q_\rho(x))}{\rho^d}. \end{aligned}$$

To verify the last inequality in (5.9), we write by using Lemma 6.2

$$\begin{aligned} m_\varepsilon(u_x; Q_\rho(x)) &= m_\varepsilon(u_x; Q_{s\rho}(x)) + m_\varepsilon(u_x; \Sigma_\rho(x)) \\ &\geq m_\varepsilon(u_x; Q_{s\rho}(x)), \end{aligned}$$

dividing by ρ^d and taking the $\overline{\lim}$ as $\varepsilon \rightarrow 0$ we have

$$\frac{m_+(u_x; Q_\rho(x))}{\rho^d} \geq s^d \frac{m_+(u_x; Q_{s\rho}(x))}{(s\rho)^d}.$$

Passing to the limits $\rho \rightarrow 0$ and then $s \rightarrow 1$ we obtain the last inequality in (5.9).

So, it remains to prove (5.8). Using (P₁) we have for some $C > 0$ dependent on p only

$$\begin{aligned} &\frac{m_\varepsilon(w; \Sigma_\rho(x))}{\rho^d} \\ &\leq \beta \left(\frac{\nu(\Sigma_\rho(x))}{\rho^d} + \frac{1}{\rho^d} \int_{\Sigma_\rho(x)} |\phi \nabla u(x) + (1-\phi) \nabla u + \nabla \phi \otimes (u_x - u)|^p dy \right) \\ &\quad + \frac{\beta}{\rho^d} \int_{\Sigma_\rho(x)} |\phi u_x + (1-\phi) u|^p dy \\ &\leq C\beta \left(\frac{\nu(\Sigma_\rho(x))}{\rho^d} + (1-s^d) |\nabla u(x)|^p + \int_{Q_\rho(x)} |\nabla u|^p dy - s^d \int_{Q_{s\rho}(x)} |\nabla u|^p dy \right) \\ &\quad + C\beta \left(\frac{4^p}{(1-s)^p} \left(\frac{1}{\rho^p} \int_{Q_\rho(x)} |u_x - u|^p dy - \frac{s^{d+p}}{(s\rho)^p} \int_{Q_{s\rho}(x)} |u_x - u|^p dy \right) \right) \\ &\quad + C\beta \rho^p \left(\frac{1}{\rho^p} \int_{Q_\rho(x)} |u_x - u|^p dy - \frac{s^{d+p}}{(s\rho)^p} \int_{Q_{s\rho}(x)} |u_x - u|^p dy \right) \\ &\quad + C\beta \left(\int_{Q_\rho(x)} |u|^p - s^d \int_{Q_{s\rho}(x)} |u|^p \right). \end{aligned}$$

Taking (5.2), (5.3), (5.4) and (5.5) into account and passing to the limits $\varepsilon \rightarrow 0$ then $\rho \rightarrow 0$ we obtain

$$\overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(w; \Sigma_\rho(x))}{\rho^d} \leq C\beta (1-s^d) (D_\lambda \nu(x) + |u(x)|^p + |\nabla u(x)|^p).$$

Letting $s \rightarrow 1$ we obtain (5.8).

Let us prove (H₂) now. Consider a sequence $\{\varphi_\varepsilon\}_\varepsilon \subset W^{1,p}(\Omega; \mathbb{R}^m)$ such that $\varphi_\varepsilon \rightarrow 0$ in $L^p(\Omega; \mathbb{R}^m)$ as $\varepsilon \rightarrow 0$ and satisfying $\sup_{\varepsilon > 0} F_\varepsilon(u + \varphi_\varepsilon; \Omega) < \infty$. Set $\mu_\varepsilon(\cdot) := F_\varepsilon(u + \varphi_\varepsilon; \cdot)$ for any $\varepsilon > 0$. There exists a subsequence (not relabeled) and a nonnegative Radon measure μ_0 such that

$$(5.10) \quad \mu_\varepsilon \xrightarrow{*} \mu_0.$$

Fix $\varepsilon > 0, s \in]1, 2[$ and $\rho > 0$. Fix $x \in O$ such that (5.2), (5.3), (5.4) and (5.5) hold and

$$(5.11) \quad D_\lambda \mu_0(x) := \lim_{r \rightarrow 0} \frac{\mu_0(Q_r(x))}{r^d} < \infty.$$

Let $\phi \in W_0^{1,\infty}(Q_{s\rho}(x); [0, 1])$ be a cut-off function between $\overline{Q}_\rho(x)$ and $Q_{s\rho}(x)$ such that

$$\|\nabla \phi\|_{L^\infty(Q_{s\rho}(x))} \leq \frac{4}{\rho(s-1)}.$$

Let $v_\varepsilon \in (u + \varphi_\varepsilon) + W_0^{1,p}(Q_\rho(x); \mathbb{R}^m)$ be such that

$$(5.12) \quad F_\varepsilon(v_\varepsilon; Q_\rho(x)) \leq \varepsilon \rho^d + m_\varepsilon(u + \varphi_\varepsilon; Q_\rho(x)).$$

Set $w := \phi v_\varepsilon + (1-\phi)u_x$, we have $w \in u_x + W_0^{1,p}(Q_{s\rho}(x); \mathbb{R}^m)$ and

$$\nabla w := \begin{cases} \nabla v_\varepsilon & \text{in } Q_\rho(x) \\ \phi(\nabla u + \nabla \varphi_\varepsilon) + (1-\phi)\nabla u(x) + \nabla \phi \otimes (u + \varphi_\varepsilon - u_x) & \text{in } \Sigma_\rho(x) \end{cases}$$

where $\Sigma_\rho(x) := \mathbb{Q}_{s\rho}(x) \setminus \overline{\mathbb{Q}_\rho}(x)$. We have

$$\begin{aligned}
(5.13) \quad s^d \frac{m_\varepsilon(u_x; \mathbb{Q}_{s\rho}(x))}{(s\rho)^d} &= s^d \frac{m_\varepsilon(w; \mathbb{Q}_{s\rho}(x))}{(s\rho)^d} \\
&= \frac{m_\varepsilon(w; \mathbb{Q}_\rho(x))}{\rho^d} + \frac{m_\varepsilon(w; \Sigma_\rho(x))}{\rho^d} \\
&\leq \frac{F_\varepsilon(v_\varepsilon; \mathbb{Q}_\rho(x))}{\rho^d} + \frac{m_\varepsilon(w; \Sigma_\rho(x))}{\rho^d} \\
&\leq \varepsilon + \frac{m_\varepsilon(u + \varphi_\varepsilon; \mathbb{Q}_\rho(x))}{\rho^d} + \frac{m_\varepsilon(w; \Sigma_\rho(x))}{\rho^d} \\
&\leq \varepsilon + \frac{F_\varepsilon(u + \varphi_\varepsilon; \mathbb{Q}_\rho(x))}{\rho^d} + \frac{m_\varepsilon(w; \Sigma_\rho(x))}{\rho^d}
\end{aligned}$$

since Lemma 6.2 and (5.12). We claim that (H₂) is proved if

$$(5.14) \quad \overline{\lim}_{s \rightarrow 1} \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(w; \Sigma_\rho(x))}{\rho^d} = 0.$$

Indeed, passing to the limits $\varepsilon \rightarrow 0$, $\rho \rightarrow 0$, $s \rightarrow 1$ in (5.13) we have

$$\begin{aligned}
(5.15) \quad \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{F_\varepsilon(u + \varphi_\varepsilon; \mathbb{Q}_\rho(x))}{\rho^d} &\geq \overline{\lim}_{s \rightarrow 1} \overline{\lim}_{\rho \rightarrow 0} \frac{m_-(u_x; \mathbb{Q}_{s\rho}(x))}{(s\rho)^d} \\
&\geq \overline{\lim}_{\rho \rightarrow 0} \frac{m_-(u_x; \mathbb{Q}_\rho(x))}{\rho^d}.
\end{aligned}$$

To verify the last inequality in (5.15), we write by using Lemma 6.2

$$\begin{aligned}
m_\varepsilon(u_x; \mathbb{Q}_{s\rho}(x)) &= m_\varepsilon(u_x; \mathbb{Q}_\rho(x)) + m_\varepsilon(u_x; \Sigma_\rho(x)) \\
&\geq m_\varepsilon(u_x; \mathbb{Q}_\rho(x)),
\end{aligned}$$

dividing by ρ^d and taking the $\underline{\lim}$ as $\varepsilon \rightarrow 0$ we have

$$s^d \frac{m_-(u_x; \mathbb{Q}_{s\rho}(x))}{(s\rho)^d} \geq \frac{m_-(u_x; \mathbb{Q}_\rho(x))}{\rho^d}.$$

Passing to the limit $\rho \rightarrow 0$ and then $s \rightarrow 1$ we obtain the last inequality in (5.15).

So, it remains to prove (5.14). Using (P₁) we have for some $C > 0$ dependent on p only

$$\begin{aligned}
(5.16) \quad &\frac{m_\varepsilon(w; \Sigma_\rho(x))}{\rho^d} \\
&\leq \beta \frac{1}{\rho^d} \int_{\Sigma_\rho(x)} |\phi(u + \varphi_\varepsilon) + (1 - \phi)u_x|^p dy \\
&+ C\beta \left((s^d - 1) |\nabla u(x)|^p + \frac{\nu(\Sigma_\rho(x))}{\rho^d} \right) \\
&+ C\beta \left(\frac{1}{\rho^d} \int_{\Sigma_\rho(x)} |\nabla u + \nabla \varphi_\varepsilon|^p + \frac{1}{\rho^d} \int_{\Sigma_\rho(x)} |\nabla \phi \otimes (u + \varphi_\varepsilon - u_x)|^p dy \right) \\
&\leq C\beta \left((s^d - 1) |\nabla u(x)|^p + \frac{\nu(\Sigma_\rho(x))}{\rho^d} + \frac{1}{\alpha} \frac{1}{\rho^d} F_\varepsilon(u + \varphi_\varepsilon; \Sigma_\rho(x)) \right) \\
&+ C\beta \frac{2^{3p-1}}{(s-1)^p} \left(s^{d+p} \frac{1}{(s\rho)^p} \int_{\mathbb{Q}_{s\rho}(x)} |u_x - u|^p dy - \frac{1}{\rho^p} \int_{\mathbb{Q}_\rho(x)} |u_x - u|^p dy \right) \\
&+ C\beta \frac{2^{3p-1}}{(s-1)^p} \left(s^{d+p} \frac{1}{(s\rho)^p} \int_{\mathbb{Q}_{s\rho}(x)} |\varphi_\varepsilon|^p dy - \frac{1}{\rho^p} \int_{\mathbb{Q}_\rho(x)} |\varphi_\varepsilon|^p dy \right) \\
&+ C\beta \rho^p \left(\frac{1}{\rho^p} \int_{\mathbb{Q}_\rho(x)} |u_x - u|^p dy - \frac{s^{d+p}}{(s\rho)^p} \int_{\mathbb{Q}_{s\rho}(x)} |u_x - u|^p dy \right) \\
&+ C\beta \left(\int_{\mathbb{Q}_\rho(x)} |u|^p - s^d \int_{\mathbb{Q}_{s\rho}(x)} |u|^p \right) + C\beta \frac{1}{\rho^d} \int_{\Sigma_\rho(x)} |\varphi_\varepsilon|^p dy.
\end{aligned}$$

Using (5.10) and Alexandrov theorem we have

$$\begin{aligned}
\overline{\lim}_{\varepsilon \rightarrow 0} \frac{1}{\rho^d} F_\varepsilon(u + \varphi_\varepsilon; \Sigma_\rho(x)) &= \overline{\lim}_{\varepsilon \rightarrow 0} \frac{\mu_\varepsilon(\Sigma_\rho(x))}{\rho^d} \\
&\leq \overline{\lim}_{\varepsilon \rightarrow 0} \frac{\mu_\varepsilon(\overline{\Sigma}_\rho(x))}{\rho^d} \\
&\leq \frac{\mu_0(\overline{\Sigma}_\rho(x))}{\rho^d} \\
&\leq s^d \frac{\mu_0(\overline{Q}_{s\rho}(x))}{(s\rho)^d} - \frac{\mu_0(Q_\rho(x))}{\rho^d}.
\end{aligned}$$

Letting $\rho \rightarrow 0$ we deduce by using (5.11)

$$(5.17) \quad \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{1}{\rho^d} F_\varepsilon(u + \varphi_\varepsilon; \Sigma_\rho(x)) \leq (s^d - 1) D_\lambda \mu_0(x).$$

Taking (5.2), (5.3), (5.4), (5.5) and (5.17) into account and passing to the limits $\varepsilon \rightarrow 0$ then $\rho \rightarrow 0$ in (5.16) we obtain

$$\overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(w; \Sigma_\rho(x))}{\rho^d} \leq C\beta (s^d - 1) (D_\lambda \nu(x) + |u(x)|^p + |\nabla u(x)|^p + D_\lambda \mu_0(x))$$

since $\varphi_\varepsilon \rightarrow 0$ in $L^p(\Omega; \mathbb{R}^m)$ as $\varepsilon \rightarrow 0$. Passing to the limit $s \rightarrow 1$ we finally proved (5.14). \blacksquare

As an illustration of Theorem 5.1 we give two elementary examples.

Example 5.1 (Integrands ‘‘almost’’ nondecreasing). *For each $\varepsilon > 0$ we consider $L_\varepsilon : \Omega \times \mathbb{R}^d \times \mathbb{M}^{m \times d} \rightarrow [0, \infty[$ a Borel measurable function such that*

$$(P_2) \quad \exists \gamma \geq 0 \quad \exists \delta > 0 \quad \forall \varepsilon > 0 \quad \forall \eta \in]0, \varepsilon[\quad \forall (x, v, \xi) \in \Omega \times \mathbb{R}^d \times \mathbb{M}^{m \times d}$$

$$L_\varepsilon(x, v, \xi) \leq L_\eta(x, v, \xi) + \gamma |\varepsilon - \eta|^\delta.$$

Note that if $\gamma = 0$ then $\varepsilon \mapsto L_\varepsilon(\cdot, \cdot, \cdot)$ is nondecreasing when ε is decreasing.

We define $F_\varepsilon : L^p(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \rightarrow [0, \infty]$ by

$$F_\varepsilon(u; O) := \int_O L_\varepsilon(x, u(x), \nabla u(x)) dx.$$

Then it is direct to see that (H_1) holds. If we assume (5.1) then (P_1) holds.

Example 5.2 (Constant integrands with perturbation). *Let $W : \Omega \times \mathbb{R}^d \times \mathbb{M}^{m \times d} \rightarrow [0, \infty[$ be a Borel measurable integrand satisfying p -growth and p -coercivity (5.1). Let $\{\Phi_\varepsilon\}_\varepsilon \subset L^1(\Omega; \mathbb{R}^+)$ such that*

(i) *there exists $g \in L^1(\Omega)$ such that $\Phi_\varepsilon(x) \leq g(x)$ for all $x \in \Omega$ and all $\varepsilon > 0$;*

(ii) *there exists a nonnegative Borel measure Φ_0 such that*

$$(5.18) \quad \Phi_\varepsilon(\cdot) \lambda \xrightarrow{*} \Phi_0 \quad \text{as } \varepsilon \rightarrow 0.$$

For each $\varepsilon > 0$ we set $L_\varepsilon : \Omega \times \mathbb{R}^d \times \mathbb{M}^{m \times d} \rightarrow [0, \infty[$ defined by

$$L_\varepsilon(x, v, \xi) = W(x, v, \xi) + \Phi_\varepsilon(x).$$

Then for each $O \in \mathcal{O}(\Omega)$ the family $\mathcal{F}(\cdot; O) \Gamma(L^p)$ -converges to

$$\mathcal{F}_0(u; O) = \int_O W_0(x, u(x), \nabla u(x)) + D_\lambda \Phi_0(x) dx.$$

Indeed, we have that (P_1) holds because of the p -growth of W and (i). Now, we have for almost all $x \in \Omega$

$$\lim_{\rho \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \int_{Q_\rho(x)} \Phi_\varepsilon(y) dy = D_\lambda \Phi_0(x)$$

since (5.18). We can see that for every $x \in \Omega$, every $\varepsilon > 0$, every $\rho > 0$ and every $u \in W^{1,p}(\Omega; \mathbb{R}^m)$

$$\inf \left\{ \int_{Q_\rho(x)} W(y, w(y), \nabla w(y)) dy : w \in u_x + W_0^{1,p}(Q_\rho(x); \mathbb{R}^m) \right\} + \int_{Q_\rho(x)} \Phi_\varepsilon(y) dy.$$

It means that (\mathcal{H}) holds and Theorem 5.2 applies with

$$L_0(x, u(x), \nabla u(x)) = W_0(x, u(x), \nabla u(x)) + D_\lambda \Phi_0(x).$$

We give a concrete example. Assume that $\Omega = B_1(0) \subset \mathbb{R}^d$ the euclidean open ball with center 0 and radius 1. Let $g : \Omega \rightarrow [0, \infty]$ be defined by

$$g(x) := \begin{cases} \frac{2}{\sqrt{\|x\|}} & \text{if } x \in \Omega \setminus \{0\} \\ +\infty & \text{if } x = 0. \end{cases}$$

where $\|\cdot\|$ is the euclidean norm. Then $g \in L^1(\Omega)$. For each $\varepsilon > 0$ we set for every $x \in \Omega$

$$\Phi_\varepsilon(x) := \frac{1}{\sqrt{\varepsilon}} \mathbb{1}_{B_\varepsilon(0)}(x) + h(x)$$

where $h \in L^1(\Omega)$ and satisfies $h(x) \leq \frac{1}{2}g(x)$ for all $x \in \Omega$. Then (i) and (ii) hold with

$$\Phi_\varepsilon(\cdot) \lambda \xrightarrow{*} \Phi_0 := \delta_0 + h\lambda \quad \text{as } \varepsilon \rightarrow 0$$

where δ_0 is the dirac measure at 0. It follows that

$$D_\lambda \Phi_0(x) = h(x) \quad \text{a.e. in } \Omega.$$

5.2. Homogenization. Let $L : \mathbb{R}^d \times \mathbb{M}^{m \times d} \rightarrow [0, \infty[$ be a Borel measurable function which is p -coercive, i.e., there exists $\alpha > 0$ such that

$$\alpha|\xi|^p \leq L(x, \xi)$$

for all $(x, \xi) \in \Omega \times \mathbb{M}^{m \times d}$. For each $\varepsilon > 0$ we consider $F_\varepsilon : W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \rightarrow [0, \infty]$ given by

$$F_\varepsilon(u; O) := \int_O L\left(\frac{x}{\varepsilon}, \nabla u(x)\right) dx.$$

The family $\mathcal{F} = \{F_\varepsilon\}_\varepsilon \subset \mathcal{I}(p, \alpha)$. For each $\xi \in \mathbb{M}^{m \times d}$ we define $\mathcal{S}_\xi : \mathcal{O}(\Omega) \rightarrow [0, \infty]$ a set function by

$$\mathcal{S}_\xi^L(O) := \inf \left\{ \int_O L(y, \xi + \nabla \varphi(y)) dy : \varphi \in W_0^{1,p}(O; \mathbb{R}^m) \right\}.$$

Definition 5.1. We say that L is an H-integrand (H stands for ‘‘homogenizable’’) if

$$(\mathcal{H}) \quad \forall \xi \in \mathbb{M}^{m \times d} \quad \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{t \rightarrow \infty} \frac{\mathcal{S}_\xi^L(tQ_\rho(x))}{\lambda(tQ_\rho(x))} = \overline{\lim}_{\rho \rightarrow 0} \underline{\lim}_{t \rightarrow \infty} \frac{\mathcal{S}_\xi^L(tQ_\rho(x))}{\lambda(tQ_\rho(x))} \quad \text{a.e. in } \Omega.$$

In this case we denote the common value by $L_{\text{hom}}(x, \xi)$.

We see that (\mathcal{H}) implies (H_1) , indeed, for every $u \in M_{\mathcal{F}}(O)$ we have

$$\frac{\mathcal{S}_{\nabla u(x)}^L\left(\frac{1}{\varepsilon}Q_\rho(x)\right)}{\lambda\left(\frac{1}{\varepsilon}Q_\rho(x)\right)} = \frac{m_\varepsilon(u_x; Q_\rho(x))}{\rho^d}.$$

for all $\varepsilon > 0$ and all $x \in O$. So, we can deduce from Theorem 5.1 the following result.

Theorem 5.2. If (P_1) holds and L is an H-integrand, i.e., (\mathcal{H}) holds. Then for each $O \in \mathcal{O}(\Omega)$ the family $\mathcal{F}(\cdot; O) \Gamma(L^p)$ -converges at every $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ to

$$\mathcal{F}_0(u, O) = \int_O L_{\text{hom}}(x, \nabla u(x)) dx$$

where

$$L_{\text{hom}}(x, \xi) = L_0(x, \xi) = \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{t \rightarrow \infty} \frac{\mathcal{S}_\xi^L(tQ_\rho(x))}{\lambda(tQ_\rho(x))}.$$

for all $x \in O$ and $\xi \in \mathbb{M}^{m \times d}$.

Theorem 5.2 becomes a “classical” homogenization result when L_{hom} does not depend on x . For instance, when L is 1-periodic or almost periodic with respect to the first variable then by subadditive theorems [LM02, Theorem 2.1 and Theorem 3.1] the condition (\mathcal{H}) holds, i.e., L is an H-integrand, and we have

$$(5.19) \quad L_{\text{hom}}(\xi) = \inf_{n \in \mathbb{N}} \frac{\mathcal{S}_{\xi}^L(nY)}{n^d} \quad (\text{periodic case})$$

$$(5.20) \quad L_{\text{hom}}(\xi) = \lim_{n \rightarrow \infty} \frac{\mathcal{S}_{\xi}^L(nY)}{n^d} \quad (\text{almost-periodic case}).$$

Example 5.3 (Periodic integrand with perturbation). *Consider $W : \mathbb{R}^d \times \mathbb{M}^{m \times d} \rightarrow [0, \infty[$ be a Borel measurable function 1-periodic with respect to the first variable, i.e.,*

$$\forall x \in \mathbb{R}^d \quad \forall z \in \mathbb{Z}^d \quad \forall \xi \in \mathbb{M}^{m \times d} \quad W(x+z, \xi) = W(x, \xi),$$

and satisfying p -growth and p -coercivity, i.e., there exist $\alpha, \beta > 0$ such that

$$\forall (x, \xi) \in \mathbb{R}^d \times \mathbb{M}^{m \times d} \quad \alpha|\xi|^p \leq W(x, \xi) \leq \beta(1 + |\xi|^p).$$

Let $\Phi \in L_{\text{loc}}^1(\mathbb{R}^d; \mathbb{R}^+)$ such that

(i) there exists $g \in L_{\text{loc}}^1(\mathbb{R}^d)$ such that $\Phi\left(\frac{x}{\varepsilon}\right) \leq g(x)$ for all $x \in \Omega$ and all $\varepsilon > 0$;

(ii) there exists a nonnegative Borel measure Φ_0 such that

$$(5.21) \quad \Phi\left(\frac{\cdot}{\varepsilon}\right) \lambda \xrightarrow{*} \Phi_0 \quad \text{as } \varepsilon \rightarrow 0.$$

Let $L : \mathbb{R}^d \times \mathbb{M}^{m \times d} \rightarrow [0, \infty[$ be defined by

$$L(x, \xi) = W(x, \xi) + \Phi(x).$$

Note that L is not periodic with respect to the first variable, because of the “perturbation” Φ .

We consider the family $\mathcal{F} = \{F_{\varepsilon}\}_{\varepsilon} \subset \mathcal{I}(p, \alpha)$ given by

$$F_{\varepsilon}(u; O) := \int_O L\left(\frac{x}{\varepsilon}, \nabla u(x)\right) dx$$

for all $(u, O) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega)$. Then $\mathcal{F}(\cdot; O)$ $\Gamma(L^p)$ -converges to

$$\mathcal{F}_0(u; O) = \int_O W_{\text{hom}}(\nabla u(x)) + D_{\lambda}\Phi_0(x) dx$$

for all $u \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega)$, and where $W_{\text{hom}}(\xi)$ is given by the formula (5.19) with \mathcal{S}_{ξ}^W in place of \mathcal{S}_{ξ}^L . Indeed, (P_1) holds because of the p -growth of W and (i). Now, we have for almost all $x \in \Omega$

$$\lim_{\rho \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \int_{Q_{\rho}(x)} \Phi\left(\frac{y}{\varepsilon}\right) dy = D_{\lambda}\Phi_0(x)$$

since (5.21). Using [LM02, Theorem 2.1] we have for every $\xi \in \mathbb{M}^{m \times d}$

$$W_{\text{hom}}(\xi) + D_{\lambda}\Phi_0(x) = \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{t \rightarrow \infty} \frac{\mathcal{S}_{\xi}^L(tQ_{\rho}(x))}{\lambda(tQ_{\rho}(x))} = \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{t \rightarrow \infty} \frac{\mathcal{S}_{\xi}^L(tQ_{\rho}(x))}{\lambda(tQ_{\rho}(x))} \quad \text{a.e. in } \Omega,$$

since we can see that for every $x \in \Omega$, every $t > 0$ and every $\rho > 0$

$$\begin{aligned} \frac{\mathcal{S}_{\xi}^L(tQ_{\rho}(x))}{\lambda(tQ_{\rho}(x))} &= \inf \left\{ \int_{tQ_{\rho}(x)} W(y, \xi + \nabla \varphi(y)) dy : \varphi \in W_0^{1,p}(tQ_{\rho}(x); \mathbb{R}^m) \right\} \\ &\quad + \int_{tQ_{\rho}(x)} \Phi(y) dy. \end{aligned}$$

It means that L is an H-integrand and Theorem 5.2 apply with

$$L_{\text{hom}}(x, \xi) = W_{\text{hom}}(\xi) + D_{\lambda}\Phi_0(x).$$

Remark 5.1. An interesting problem in the field of deterministic homogenization (see [NNW10]) is the characterization of all H-integrands $L : \mathbb{R}^d \times \mathbb{M}^{m \times d} \rightarrow [0, \infty[$ Borel measurable with p -growth and p -coercivity, i.e., satisfying

$$\begin{aligned} \exists \alpha > 0 \quad \exists \beta > 0 \quad \forall (x, \xi) \in \Omega \times \mathbb{M}^{m \times d} \\ \alpha |\xi|^p \leq L(x, \xi) \leq \beta (1 + |\xi|^p). \end{aligned}$$

5.3. Relaxation. The following result is an extension of Acerbi-Fusco-Dacorogna relaxation theorem (see [Dac08, Theorem 9.8, p. 432] and [AF84, Statement III.7, p. 144]) in the case where the integrand is assumed Borel measurable only.

Theorem 5.3. *If $L : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m \times d} \rightarrow [0, \infty[$ is Borel measurable and satisfies (5.1) then for every $O \in \mathcal{O}(\Omega)$*

$$\mathcal{F}_0(u; O) = \int_O L_0(x, u(x), \nabla u(x)) dx$$

where for a.a. $x \in O$

$$\begin{aligned} L_0(x, u(x), \nabla u(x)) \\ = \liminf_{\rho \rightarrow 0} \left\{ \int_{Q_\rho(x)} L(y, w(y), \nabla w(y)) dy : w \in u_x + W_0^{1,p}(Q_\rho(x); \mathbb{R}^m) \right\}. \end{aligned}$$

Moreover, if L is Carathéodory, i.e.,

- (i) for each $(v, \xi) \in \mathbb{R}^m \times \mathbb{M}^{m \times d}$ the function $\Omega \ni x \mapsto L(x, v, \xi)$ is measurable;
- (ii) for a.a. $x \in \Omega$ the function $\mathbb{R}^m \times \mathbb{M}^{m \times d} \ni (v, \xi) \mapsto L(x, v, \xi)$ is continuous,

then for almost every $x \in \Omega$ and for every $(v, \xi) \in \mathbb{R}^d \times \mathbb{M}^{m \times d}$

$$(5.22) \quad \tilde{L}_0(x, v, \xi) = \inf \left\{ \int_Y L(x, v, \xi + \nabla \varphi(y)) dy : \varphi \in W_0^{1,\infty}(Y; \mathbb{R}^m) \right\}.$$

Proof. The formula (5.22) follows from Proposition 5.1. \blacksquare

Remark 5.2. Under the same assumptions of Theorem 5.3 and using Proposition 2.2 we also have

$$\mathcal{F}_0(u; O) = \mathcal{F}_0^{\mathfrak{D}}(u; O) = \int_O L_0(x, u(x), \nabla u(x)) dx$$

for all $(u, O) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega)$.

We can give an extension of $W^{1,p}$ -quasiconvexity as follows.

Definition 5.2. *We say that a Borel measurable integrand $L : \Omega \times \mathbb{R}^d \times \mathbb{M}^{m \times d} \rightarrow [0, \infty[$ is $W^{1,p}$ -quasiconvex if for every $(x, v, \xi) \in \Omega \times \mathbb{R}^d \times \mathbb{M}^{m \times d}$*

$$\tilde{L}_0(x, v, \xi) = L(x, v, \xi).$$

However, when the integrand is dependent on (x, v) this generalization of quasiconvexity is more difficult to handle. When the integrand L is Carathéodory the variables x and v can be frozen and we recover the classical concept of quasiconvexity.

Proposition 5.1. *If L is Carathéodory and satisfies p -growth (5.1) then for a.a. $x \in \Omega$ and for every $(v, \xi) \in \mathbb{R}^m \times \mathbb{M}^{m \times d}$ we have*

$$(5.23) \quad \tilde{L}_0(x, v, \xi) = \inf \left\{ \int_Y L(x, v, \xi + \nabla \varphi(y)) dy : \varphi \in W_0^{1,\infty}(Y; \mathbb{R}^m) \right\}.$$

Proof. For each $(x, v, \xi) \in \Omega \times \mathbb{R}^m \times \mathbb{M}^{m \times d}$ we denote by $Q^{\text{dac}}L(x, v, \xi)$ the right hand side of (5.23). For each $\rho \in]0, 1[$ we define $\Lambda_\rho, L_\rho : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m \times d} \rightarrow [0, \infty[$ by

$$\begin{aligned} \Lambda_\rho(x, v, \xi) &:= \inf \left\{ \int_Y L(x + \rho y, v + \rho(\xi y + \psi(y)), \xi + \nabla \psi(y)) dy : \right. \\ &\quad \left. \psi \in W_0^{1, \infty}(Y; \mathbb{R}^m) \right\}; \\ L_\rho(x, v, \xi) &:= \inf \left\{ \int_Y L(x + \rho y, v + \rho(\xi y + \varphi(y)), \xi + \nabla \varphi(y)) dy : \right. \\ &\quad \left. \varphi \in W_0^{1, p}(Y; \mathbb{R}^m) \right\}. \end{aligned}$$

It is easy to see, by a change of variables, that for a.a. $x \in \Omega$ and for every $(v, \xi) \in \mathbb{R}^m \times \mathbb{M}^{m \times d}$ we have

$$(5.24) \quad \overline{\lim}_{\rho \rightarrow 0} L_\rho(x, v, \xi) = \tilde{L}_0(x, v, \xi).$$

It is enough to show that for a.a. $x \in \Omega$, for every $(v, \xi) \in \mathbb{R}^m \times \mathbb{M}^{m \times d}$ and every $\rho \in]0, 1[$ it hold

$$(5.25) \quad Q^{\text{dac}}L(x, v, \xi) = \lim_{\rho \rightarrow 0} \Lambda_\rho(x, v, \xi)$$

$$(5.26) \quad \Lambda_\rho(x, v, \xi) = L_\rho(x, v, \xi).$$

Indeed, combining (5.24), (5.25) and (5.26) we obtain (5.23).

Proof of (5.25). Let $\delta > 0$. By Scorza-Dragnoni theorem, there exists a compact set $K_\delta \subset \bar{Y}$ such that $\lambda(Y \setminus K_\delta) < \delta$ and $L|_{K_\delta \times (\mathbb{R}^m \times \mathbb{M}^{m \times d})}$ is continuous. Fix $(x, v, \xi) \in \Omega \times \mathbb{R}^m \times \mathbb{M}^{m \times d}$ such that

$$(5.27) \quad a(x) = \lim_{\rho \rightarrow 0} \int_{Q_\rho(x)} a(y) dy = \lim_{\rho \rightarrow 0} \int_Y a(x + \rho y) dy < \infty.$$

We show first that $\overline{\lim}_{\rho \rightarrow 0} \Lambda_\rho(x, v, \xi) \leq Q^{\text{dac}}L(x, v, \xi)$. Note that

$$Q^{\text{dac}}L(x, v, \xi) \leq L(x, v, \xi) \leq \beta(a(x) + |v|^p + |\xi|^p) < \infty.$$

Let $\varepsilon > 0$. There exists $\psi \in W_0^{1, \infty}(Y; \mathbb{R}^m)$ such that

$$(5.28) \quad \int_Y L(x, v, \xi + \nabla \psi(y)) dy \leq \varepsilon + Q^{\text{dac}}L(x, v, \xi).$$

Fix $\rho \in]0, 1[$. Set $g_\rho(y) := L(x + \rho y, v + \rho(\xi y + \psi(y)), \xi + \nabla \psi(y))$ and $g_0(y) := L(x, v, \xi + \nabla \psi(y))$ for all $y \in Y$. Using (5.28) we have

$$\begin{aligned} (5.29) \quad \Lambda_\rho(x, v, \xi) &\leq \int_{K_\delta} g_\rho(y) dy + \int_{Y \setminus K_\delta} g_\rho(y) dy \\ &= \int_{K_\delta} g_\rho(y) - g_0(y) dy + \int_{Y \setminus K_\delta} g_\rho(y) - g_0(y) dy \\ &\quad + \int_Y g_0(y) dy \\ &\leq \int_{K_\delta} |g_\rho(y) - g_0(y)| dy + \int_{Y \setminus K_\delta} |g_\rho(y) - g_0(y)| dy \\ &\quad + \varepsilon + Q^{\text{dac}}L(x, v, \xi). \end{aligned}$$

By using the p -growth (5.1) it is easy to see that there exists C depending on β and p only such that

$$(5.30) \quad \max\{g_0(y), g_\rho(y)\} \leq C(a(x + \rho y) + |v|^p + |\xi|^p + |\psi(y)|^p + |\nabla \psi(y)|^p) \text{ a.e. in } Y.$$

By continuity of $L|_{K_\delta \times (\mathbb{R}^m \times \mathbb{M}^{m \times d})}$ we have $g_\rho(y) - g_0(y) \rightarrow 0$ a.e. in K_δ as $\rho \rightarrow 0$. Using the domination (5.30) we obtain by applying the Lebesgue dominated convergence theorem

$$(5.31) \quad \lim_{\rho \rightarrow 0} \int_{K_\delta} |g_\rho(y) - g_0(y)| dy = 0.$$

By (5.30) we have

$$(5.32) \quad \int_{Y \setminus K_\delta} |g_\rho(y) - g_0(y)| dy \leq 2C \left(\int_{Y \setminus K_\delta} a(x + \rho y) dy + \delta (a(x) + |v|^p + |\xi|^p + \|\psi\|_\infty^p + \|\nabla \psi\|_\infty^p) \right).$$

Note that $\{Y \ni y \mapsto a(x + \rho y)\}_{\rho \in]0,1[}$ is uniformly integrable since (5.27). So, taking the supremum over ρ and passing to the limit $\delta \downarrow 0$ in (5.32) we find that

$$(5.33) \quad \lim_{\delta \downarrow 0} \sup_{\rho \in]0,1[} \int_{Y \setminus K_\delta} |g_\rho(y) - g_0(y)| dy = 0.$$

Taking (5.31) and (5.33) into account in (5.29) we find

$$\varliminf_{\rho \rightarrow 0} \Lambda_\rho(x, v, \xi) \leq \varepsilon + Q^{\text{dac}} L(x, v, \xi).$$

Now, we want to show that $\varliminf_{\rho \rightarrow 0} \Lambda_\rho(x, v, \xi) \geq Q^{\text{dac}} L(x, v, \xi)$. Consider a sequence $\{\rho_n\}_{n \in \mathbb{N}} \subset]0, 1[$ such that

$$\varliminf_{\rho \rightarrow 0} \Lambda_\rho(x, v, \xi) = \lim_{n \rightarrow \infty} \Lambda_{\rho_n}(x, v, \xi) \leq \beta (a(x) + |v|^p + |\xi|^p) < \infty$$

since p -growth conditions (5.1). Fix $n \in \mathbb{N}$. We can choose $\psi_n \in W_0^{1,\infty}(Y; \mathbb{R}^m)$ such that

$$\int_Y g_n(y) dy \leq \rho_n + \Lambda_{\rho_n}(x, v, \xi)$$

where $g_n(y) := L(x + \rho_n y, v + \rho_n(\xi y + \psi_{\rho_n}(y)), \xi + \nabla \psi_{\rho_n}(y))$ for all $y \in Y$. Since p -coercivity, we can choose a subsequence (not relabelled) such that

$$(5.34) \quad \psi_n \rightarrow \psi_\infty \text{ in } L^p(Y; \mathbb{R}^m);$$

$$(5.35) \quad \nabla \psi_n \rightarrow \nabla \psi_\infty \text{ in } L^p(Y; \mathbb{M}^{m \times d}).$$

Fix $\delta > 0$ and choose a compact set $K_\delta \subset \bar{Y}$ such that $\lambda(Y \setminus K_\delta) < \delta$ and $L|_{[K_\delta \times (\mathbb{R}^m \times \mathbb{M}^{m \times d})]}$ is continuous. We have by Eisen convergence theorem [Eis79, p. 75] that

$$(5.36) \quad g_n(y) - L(x, v, \xi + \nabla \psi_n(y)) \rightarrow 0 \text{ in measure in } K_\delta.$$

We have

$$\begin{aligned} \int_Y g_n(y) dy &\geq \int_{K_\delta} g_n(y) - L(x, v, \xi + \nabla \psi_n(y)) dy \\ &\quad + \int_{Y \setminus K_\delta} g_n(y) - L(x, v, \xi + \nabla \psi_n(y)) dy + Q^{\text{dac}} L(x, v, \xi). \end{aligned}$$

Using growth conditions we have for a.a. $y \in Y$

$$(5.37) \quad \begin{aligned} &|g_n(y) - L(x, v, \xi + \nabla \psi_n(y))| \\ &\leq 2C (a(x + \rho_n y) + a(x) + |v|^p + |\xi|^p + |\psi_n(y)|^p + |\nabla \psi_n(y)|^p). \end{aligned}$$

By taking (5.36), (5.37), (5.34) and (5.35) into account we have

$$\lim_{n \rightarrow \infty} \int_{K_\delta} |g_n(y) - L(x, v, \xi + \nabla \psi_n(y))| dy = 0$$

since Vitali convergence theorem. Using (5.37) and reasoning similarly as in the first part of the proof we have

$$\lim_{\delta \rightarrow 0} \sup_{n \in \mathbb{N}} \int_{Y \setminus K_\delta} |g_n(y) - L(x, v, \xi + \nabla \psi_n(y))| dy = 0.$$

It follows that

$$\varliminf_{\rho \rightarrow 0} \Lambda_\rho(x, v, \xi) = \lim_{n \rightarrow \infty} \Lambda_{\rho_n}(x, v, \xi) \geq \varliminf_{n \rightarrow \infty} \int_Y g_n(y) dy \geq Q^{\text{dac}} L(x, v, \xi).$$

Proof of (5.26). Fix $(x, v, \xi) \in \Omega \times \mathbb{R}^m \times \mathbb{M}^{m \times d}$ and $\rho \in]0, 1[$. We only need to prove that

$$(5.38) \quad L_\rho(x, v, \xi) \geq \Lambda_\rho(x, v, \xi).$$

Let $\varepsilon > 0$. There exists $\varphi_\varepsilon \in W_0^{1,p}(Y; \mathbb{R}^m)$ such that

$$L_\rho(x, v, \xi) + \varepsilon \geq \int_Y L(x + \rho y, v + \rho(\xi y + \varphi_\varepsilon(y)), \xi + \nabla \varphi_\varepsilon(y)) dy.$$

There exists a sequence $\{\psi_n\}_{n \in \mathbb{N}} \subset W_0^{1,\infty}(Y; \mathbb{R}^m)$ such that $\psi_n \rightarrow \varphi_\varepsilon$ in $W^{1,p}(Y; \mathbb{R}^m)$, $\psi_n \rightarrow \varphi_\varepsilon$ a.e. in Y and $\nabla \psi_n \rightarrow \nabla \varphi_\varepsilon$ a.e. in Y as $n \rightarrow \infty$. Using growth conditions we have for some C depending on β and p only, for a.a. $y \in Y$ and for all $n \in \mathbb{N}$

$$\begin{aligned} & L(x + \rho y, v + \rho(\xi y + \psi_n(y)), \xi + \nabla \psi_n(y)) \\ & \leq C(a(x + \rho y) + |v|^p + |\xi|^p + |\psi_n(y)|^p + |\nabla \psi_n(y)|^p). \end{aligned}$$

Since L is Carathéodory we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} L(x + \rho y, v + \rho(\xi y + \psi_n(y)), \xi + \nabla \psi_n(y)) \\ & = L(x + \rho y, v + \rho(\xi y + \varphi_\varepsilon(y)), \xi + \nabla \varphi_\varepsilon(y)) \quad \text{a.e. in } Y. \end{aligned}$$

Applying Vitali convergence theorem we obtain

$$\begin{aligned} \Lambda_\rho(x, v, \xi) & \leq \lim_{n \rightarrow \infty} \int_Y L(x + \rho y, v + \rho(\xi y + \psi_n(y)), \xi + \nabla \psi_n(y)) dy \\ & = \int_Y L(x + \rho y, v + \rho(\xi y + \varphi_\varepsilon(y)), \xi + \nabla \varphi_\varepsilon(y)) dy \\ & \leq L_\rho(x, v, \xi) + \varepsilon. \end{aligned}$$

Letting $\varepsilon \rightarrow 0$ we finally obtain (5.38). \blacksquare

6. APPENDIX

6.1. Usage of Vitali covering theorem. Let $A \subset O \in \mathcal{O}(\Omega)$ be a set which is not necessarily measurable. For each $x \in A$ we consider a family of closed balls \mathcal{K}_x containing x of O satisfying $\inf \{\text{diam}(Q) : Q \in \mathcal{K}_x\} = 0$ and $A \subset \cup_{Q \in \mathcal{K}} Q$ with $\mathcal{K} := \cup_{x \in A} \mathcal{K}_x$. We say that \mathcal{K} is a fine cover of A .

Then there exists a countable pairwise disjoint family of balls $\{\bar{Q}_i\}_{i \geq 1} \subset \mathcal{K}$ such that

$$\lambda\left(A \setminus \bigcup_{i=1}^{\infty} Q_i\right) = 0.$$

It follows that for any $\mu \in \mathfrak{A}_\lambda(O)$, i.e. $\mu \ll \lambda|_O$, we have $\mu(A \setminus \cup_{i \geq 1} Q_i) = 0$. Moreover, if $\lambda(A) < \infty$ then for any $\delta > 0$ we can choose a finite subfamily $\{\bar{Q}_i\}_{i=1}^N \subset \mathcal{K}$ satisfying

$$\mu\left(A \setminus \bigcup_{i=1}^N Q_i\right) < \delta.$$

6.2. Level sets of derivative of set functions. Let $G : \mathcal{Q}_o(\Omega) \rightarrow]-\infty, \infty]$ be a set function. Let $O \in \mathcal{O}(\Omega)$. For each $h \in \mathbb{R}$ we consider the strict sublevel (resp. superlevel) of the lower (resp. upper) derivative of G

$$S_h := \{x \in O : \underline{D}_\lambda G(x) < h\} \quad (\text{resp. } S^h := \{x \in O : \bar{D}_\lambda G(x) > h\})$$

The following lemma give consequences of sublevel (resp. superlevel) sets of derivative of set functions.

Lemma 6.1. *Let $h \in \mathbb{R}$ and $\eta > 0$. Then*

(i) *there exists a countable pairwise disjoint family $\{Q_i\}_{i \in I} \subset \mathcal{Q}_o(O)$ such that*

$$(6.1) \quad \lambda\left(S_h \setminus \bigcup_{i \in I} Q_i\right) = 0, \quad \forall i \in I \quad G(Q_i) < h\lambda(Q_i) \text{ and } \text{diam}(Q_i) \in]0, \eta[$$

(resp. $\lambda\left(S^h \setminus \bigcup_{i \in I} Q_i\right) = 0, \quad \forall i \in I \quad G(Q_i) > h\lambda(Q_i) \text{ and } \text{diam}(Q_i) \in]0, \eta[;$

(ii) *for every $\delta > 0$ there exists a finite pairwise disjoint family $\{Q_i\}_{i \in I} \subset \mathcal{Q}_o(O)$ such that*

$$\lambda\left(S_h \setminus \bigcup_{i \in I} Q_i\right) < \delta, \quad \forall i \in I \quad G(Q_i) < h\lambda(Q_i) \text{ and } \text{diam}(Q_i) \in]0, \eta[$$

(resp. $\lambda\left(S^h \setminus \bigcup_{i \in I} Q_i\right) < \delta, \quad \forall i \in I \quad G(Q_i) > h\lambda(Q_i) \text{ and } \text{diam}(Q_i) \in]0, \eta[.$

Proof. Let $h \in \mathbb{R}$ and $\eta > 0$. We only give the proof for S_h , since similar arguments apply for S^h . Note that (ii) is a direct consequence of (i), so, we only show (i).

If $x \in S_h$ then for some $\varepsilon > 0$

$$\forall \rho \in]0, \eta[\quad \inf \left\{ \frac{G(Q)}{\lambda(Q)} : Q \in \mathfrak{B}_{x,\rho}(O) \right\} < h - \varepsilon$$

where $\mathfrak{B}_{x,\rho}(O) := \{Q : x \in Q \in \mathcal{Q}_o(O) \text{ and } \text{diam}(Q) \leq \rho\}$. For each $\rho \in]0, \eta[$ there exists $Q_{x,\rho} \in \mathfrak{B}_{x,\rho}(O)$ such that

$$(6.2) \quad \frac{G(Q_{x,\rho})}{\lambda(Q_{x,\rho})} - \varepsilon \leq \inf \left\{ \frac{G(Q)}{\lambda(Q)} : Q \in \mathfrak{B}_{x,\rho}(O) \right\} < h - \varepsilon.$$

Consider the family $\mathcal{K}_\eta := \{\overline{Q_{x,\rho}}\}_{x \in S_h, \rho \in]0, \eta[}$ of closed cubes such that (6.2) holds. The family \mathcal{K}_η is a fine cover of S_h , i.e.,

$$S_h \subset \bigcup_{Q \in \mathcal{K}_\eta} Q \quad \text{and} \quad \forall x \in S_h \quad \inf \{\text{diam}(Q) : Q \in \mathcal{K}_{\eta,x}\} = 0$$

where $\mathcal{K}_{\eta,x} := \{\overline{Q_{x,\rho}}\}_{\rho \in]0, \eta[} \subset \mathcal{K}_\eta$. By Vitali covering theorem we conclude (6.1). ■

6.3. Proof of Lemma 3.1. Fix $c \in \mathbb{R}$. We have to prove that

$$M_c := \{x \in O : \underline{D}_\lambda G(x) \leq c\}$$

is measurable. Fix $\eta > 0$. Set $h := c + \eta$. By Lemma 6.1 (i) there exists a countable pairwise disjointed family $\{Q_i\}_{i \in I} \subset \mathcal{Q}_o(O)$ such that

$$\lambda \left(S_h \setminus \bigcup_{i \in I} Q_i \right) = 0, \quad \forall i \in I \quad G(Q_i) < h \lambda(Q_i) \text{ and } \text{diam}(Q_i) \in]0, \eta[.$$

Since $S_h \supset M_c$ we have

$$\lambda \left(M_c \setminus \bigcup_{i \in I} Q_i \right) = 0.$$

If we show that the Borel set $Q^\infty := \bigcup_{i \in I} Q_i \subset M_c$ then M_c will be the reunion of a Borel set and a λ -negligible set and so measurable since λ is complete. Let $z \in Q^\infty$. Then there exists $i_z \in I$ such that $z \in Q_{i_z}$. It follows that

$$\inf \left\{ \frac{G(Q)}{\lambda(Q)} : z \in Q \in \mathcal{Q}_o(O), \text{diam}(Q) \leq \eta \right\} \leq \frac{G(Q_{i_z})}{\lambda(Q_{i_z})} \leq c + \eta.$$

Passing to the limit $\eta \rightarrow 0$ we obtain $\underline{D}_\lambda G(z) \leq c$ which means that $z \in M_c$. The proof is complete. ■

6.4. Properties of the set function $\{m_\varepsilon(u; \cdot)\}_\varepsilon$.

Lemma 6.2. *Let $(u, O) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega)$. Then the family $\{m_\varepsilon(u; \cdot)\}_\varepsilon$, $m_\varepsilon(u; \cdot) : \mathcal{O}(O) \rightarrow [0, \infty]$ satisfies*

(i) *for every $\varepsilon > 0$ and every $(U, V) \in \mathcal{O}(O) \times \mathcal{O}(O)$*

$$U \cap V = \emptyset \implies m_\varepsilon(u; U \cup V) = m_\varepsilon(u; U) + m_\varepsilon(u; V);$$

(ii) *for every $\varepsilon > 0$, every $U, V \in \mathcal{O}(O)$ with $U \subset V$*

$$\lambda(V \setminus U) = 0 \implies m_\varepsilon(u; U) = m_\varepsilon(u; V);$$

(iii) *In particular, for every $U \in \mathcal{O}(O)$ and $V \in \mathcal{O}(O)$ satisfying $U \subset V$ we have for every $\varepsilon > 0$*

$$\lambda(\partial U) = 0 \implies m_\varepsilon(u; V) = m_\varepsilon(u; U) + m_\varepsilon(u; V \setminus \overline{U}).$$

Proof. We recall that for $A \in \mathcal{O}(\Omega)$ we have

$$W_0^{1,p}(A; \mathbb{R}^m) = \{u \in W^{1,p}(\Omega; \mathbb{R}^m) : u = 0 \text{ in } \Omega \setminus A\}.$$

If $U, V \in \mathcal{O}(O)$ satisfy $U \cap V = \emptyset$ then for every $\varphi_i \in L^p(\Omega; \mathbb{R}^m)$ with $i \in \{0, 1, 2\}$ we have

$$\varphi_0 \in W_0^{1,p}(U \cup V; \mathbb{R}^m) \implies \varphi_0 \mathbf{1}_U \in W_0^{1,p}(U; \mathbb{R}^m) \text{ and } \varphi_0 \mathbf{1}_V \in W_0^{1,p}(V; \mathbb{R}^m);$$

$$\varphi_1 \in W_0^{1,p}(U; \mathbb{R}^m) \text{ and } \varphi_2 \in W_0^{1,p}(V; \mathbb{R}^m) \implies \varphi_1 \mathbf{1}_U + \varphi_2 \mathbf{1}_V \in W_0^{1,p}(U \cup V; \mathbb{R}^m).$$

Let $\varepsilon > 0$. To verify (i) it suffices to write for every $\varphi_1 \in W_0^{1,p}(U; \mathbb{R}^m)$ and $\varphi_2 \in W_0^{1,p}(V; \mathbb{R}^m)$

$$\begin{aligned} F_\varepsilon(u + \varphi_1; U) + F_\varepsilon(u + \varphi_2; V) &= F_\varepsilon(u + \varphi_1 \mathbf{1}_U + \varphi_2 \mathbf{1}_V; U \cup V) \\ &\geq m_\varepsilon(u; U \cup V), \end{aligned}$$

taking the infimum over φ_1 and φ_2 we obtain

$$m_\varepsilon(u; U) + m_\varepsilon(u; V) \geq m_\varepsilon(u; U \cup V).$$

We also have for every $\varphi_0 \in W_0^{1,p}(U \cup V; \mathbb{R}^m)$

$$\begin{aligned} F_\varepsilon(u + \varphi_0; U \cup V) &= F_\varepsilon(u + \varphi_0 \mathbf{1}_U; U) + F_\varepsilon(u + \varphi_0 \mathbf{1}_V; V) \\ &\geq m_\varepsilon(u; U) + m_\varepsilon(u; V). \end{aligned}$$

Taking the infimum over φ_0 we obtain

$$m_\varepsilon(u; U \cup V) \geq m_\varepsilon(u; U) + m_\varepsilon(u; V).$$

Then (i) is satisfied.

Consider $U, V \in \mathcal{O}(O)$ satisfying $U \subset V$ and $\lambda(V \setminus U) = 0$. Since $U \subset V$ we have $W_0^{1,p}(U; \mathbb{R}^m) \subset W_0^{1,p}(V; \mathbb{R}^m)$, thus $m_\varepsilon(u; U) \geq m_\varepsilon(u; V)$. Assume that $m_\varepsilon(u; V) < \infty$. For every $\eta > 0$ there exists $\varphi \in W_0^{1,p}(V; \mathbb{R}^m)$ such that $\infty > m_\varepsilon(u; V) + \eta \geq F_\varepsilon(u + \varphi; V)$. By using (C₂) we have

$$\begin{aligned} m_\varepsilon(u; V) + \eta &\geq F_\varepsilon(u + \varphi; V) = F_\varepsilon(u + \varphi \mathbf{1}_U; U) + F_\varepsilon(u + \varphi; V \setminus U) \\ &\geq m_\varepsilon(u; U). \end{aligned}$$

Note that $\varphi \mathbf{1}_U = \varphi$ a.e. in V and so $\varphi \mathbf{1}_U \in W_0^{1,p}(U; \mathbb{R}^m)$. Therefore (ii) is satisfied.

To prove (iii) it is sufficient to use the properties (ii), (i) together with the fact that we can write $V \setminus (U \cup (V \setminus \bar{U})) = \partial U$ for all $U, V \in \mathcal{O}(O)$ satisfying $U \subset V$. ■

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