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Γ-limits of functionals determined by their minima

by

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Γ-LIMITS OF FUNCTIONALS DETERMINED BY THEIR INFIMA

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In memoriam Jean-Jacques Moreau

ABSTRACT. We study the integral representation of Γ-limits of p-coercive integral functionals of the calculus of variations in the spirit of [DMM86a]. We use infima of local Dirichlet problems to characterize the limit integrands. Applications to homogenization and relaxation are given.

1. Introduction

Let $m, d \geqslant 1$ be two integers. Let $\Omega \subset \mathbb{R}^d$ be a nonempty bounded open set with Lipschitz boundary. Let $\mathcal{O}(\Omega)$ be the class of all open subsets of Ω. We consider a family of functionals $\mathcal{F} := \{F_\varepsilon\}_{\varepsilon\in]0,1]}$ with $F_\varepsilon : W^{1,p}(\Omega;\mathbb{R}^m) \times \mathcal{O}(\Omega) \to [0,\infty]$. We set conditions in order that each functional of the family $\mathcal{F}$ can be considered as a p-coercive integral functional of the calculus of variations (see the "global" conditions (C_1), (C_2), (C_3) in Sect. 2). We are interested in the integral representation of the $\Gamma(L^p)$-limit of $\mathcal{F}$. This is an important problem in the field of Γ-convergence theory (see for instance [DG79]).

Our goal is to study the conditions of the integral representation of $\Gamma(L^p)$-limit by using the infima of local Dirichlet problems associated to $\mathcal{F}$ as in [DMM86a, BFM98, BFLM02]. More precisely, we consider the behavior of

$$m_\varepsilon(u;O) := \inf\left\{F_\varepsilon(v;O) : v \in u + W_0^{1,p}(O;\mathbb{R}^m)\right\}$$

in order to find the conditions for the integral representation (see also [DMM86b, DMM86c, Mod86]). We propose three "local" conditions (see (H_1), (H_2) and (H_3) in Sect. 2) related to the local behavior of m_ε which allows to prove $\Gamma(L^p)$-convergence of the family $\mathcal{F}(\cdot;O)$ with integral representation of the $\Gamma(L^p)$-limit $\mathcal{F}_0(\cdot;O)$

$$\mathcal{F}_0(u;O) = \int_O L_0(x, u(x), \nabla u(x))\,dx$$

where $u \in \mathrm{M}_{\mathcal{F}}(O)$ (see Definition (2.1) for $\mathrm{M}_{\mathcal{F}}(O)$) and

$$L_0(x, u(x), \nabla u(x)) = \lim_{\rho\to 0}\overline{\lim_{\varepsilon\to 0}}\frac{m_\varepsilon(u_x;\mathrm{Q}_\rho(x))}{\rho^d} \qquad \text{with } u_x(\cdot) = u(x) + \nabla u(x)(\cdot - x).$$

The main difficulty is to obtain an upper bound under integral form for the $\Gamma(L^p)$-$\overline{\lim}$. More precisely, we show, in Sect. 3 together with Sect. 4, that the Vitali envelope (which is an envelope of Carathéodory type where the arbitrary coverings are replaced by Vitali coverings) $V_+(u;\cdot)$ of the set function $\mathcal{O}(\Omega) \ni V \mapsto \overline{\lim}_{\varepsilon\to 0} m_\varepsilon(u;V)$ when $u \in \mathrm{M}_{\mathcal{F}}(O)$ satisfies

$$\Gamma(L^p)\text{-}\overline{\lim_{\varepsilon\to 0}} F_\varepsilon(u;O) \leqslant V_+(u;O) = \int_O \lim_{\rho\to 0}\inf\left\{\overline{\lim_{\varepsilon\to 0}}\frac{m_\varepsilon(u;\mathrm{Q})}{\lambda(\mathrm{Q})} : x \in \mathrm{Q} \in \mathcal{Q}_o(O),\ \mathrm{diam}(\mathrm{Q}) \leqslant \rho\right\}dx.$$

The Vitali envelope of a set function in connection with the integral representation of $\Gamma(L^p)$-limits was introduced in [BFM98] (see also [BB00]). This path has the advantage to avoid any approximations of Sobolev functions by regular ones. It allows, when we assume p-growth conditions, to give general results for $\Gamma(L^p)$-limit and in particular to give a general point of view in homogenization and relaxation problems for Borel measurable integrands $L(x, v, \xi)$ (see Sect. 5).

Plan of the paper. Sect. 2 presents the main assumptions ("global and local" conditions) and the statement of the general results (see Theorem 2.1 and Theorem 2.2). Theorem 2.2 is an integral representation result of $\Gamma(L^p)$-limit, it is a consequence of local conditions ((H_1), (H_2) and (H_3)) and Theorem 2.1. In Sect. 3 we state and prove an integral representation for the Vitali envelope of arbitrary nonnegative set functions. In Sect. 4 we give the proof of Theorem 2.1 and some other related results. Finally in Sect. 5 we give a general $\Gamma(L^p)$-convergence result in the p-growth case Theorem 5.1, which can be seen as an

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extension in a nonconvex (an vectorial) case of Theorem IV in [DMM86a, p. 265]. In fact, we show how to verify the local conditions (H_2) and (H_3) when we deal with p-growth, the technics we use are inspired by [BFM98]. In Subsect. 5.2 as an application of Theorem 5.1 we consider a general point of view of the homogenization of functional integral of the calculus of variations. In Subsect. 5.3 we give an extension of the Acerbi-Fusco-Dacorogna relaxation theorem when the integrand is assumed Borel measurable only.

2. Main results

2.1. **General framework.** Fix $\alpha > 0$ and $p \in]1, \infty[$. We denote by $\mathcal{I}(p,\alpha)$ the set of functionals $F : W^{1,p}(\Omega;\mathbb{R}^m) \times \mathcal{O}(\Omega) \to [0,\infty]$ satisfying:

- (C_1) for every $O \in \mathcal{O}(\Omega)$ and every $u \in \mathrm{dom}F(\cdot;O)$ we have
$$F(u;O) \geqslant \alpha \|\nabla u\|^p_{L^p(O;\mathbb{R}^m)};$$
- (C_2) for every $u \in \mathrm{dom}F(\cdot;\Omega)$ the set function $F(u;\cdot)$ is the trace on $\mathcal{O}(\Omega)$ of a Borel measure absolutely continuous with respect to the Lebesgue measure λ on Ω;
- (C_3) for every $O \in \mathcal{O}(\Omega)$ the functional $F(\cdot;O)$ is local, i.e., if $u = v$ a.e. in O then $F(u;O) = F(v;O)$ for all $u, v \in \mathrm{dom}F(\cdot;O)$.

Consider a family $\mathcal{F} := \{F_\varepsilon\}_{\varepsilon\in]0,1]}$ of functionals $F_\varepsilon : W^{1,p}(\Omega;\mathbb{R}^m) \times \mathcal{O}(\Omega) \to [0,\infty]$. For each $O \in \mathcal{O}(\Omega)$ and each $u \in L^p(\Omega;\mathbb{R}^m)$ we set
$$\mathcal{F}_-(u;O) := \inf\left\{\varliminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon;O) : u_\varepsilon \to u \text{ in } L^p(\Omega;\mathbb{R}^m)\right\};$$
$$\mathcal{F}_+(u;O) := \inf\left\{\varlimsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon;O) : u_\varepsilon \to u \text{ in } L^p(\Omega;\mathbb{R}^m)\right\}.$$
The functional $\mathcal{F}_-(\cdot;O)$ (resp. $\mathcal{F}_+(\cdot;O)$) is the $\Gamma(L^p)$-$\varliminf_{\varepsilon\to 0}$ (resp. $\Gamma(L^p)$-$\varlimsup_{\varepsilon\to 0}$) of the family $\mathcal{F}(\cdot;O) = \{F_\varepsilon(\cdot;O)\}_\varepsilon$. If $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ and $\mathcal{F}_+(u;O) = \mathcal{F}_-(u;O)$ then we say that $\mathcal{F}(\cdot;O)$ $\Gamma(L^p)$-converges at u to the $\Gamma(L^p)$-limit $\mathcal{F}_0(u;O) := \mathcal{F}_+(u;O) = \mathcal{F}_-(u;O)$.

We associate to $\mathcal{F} = \{F_\varepsilon\}_{\varepsilon\in]0,1]} \subset \mathcal{I}(p,\alpha)$ a family of local Dirichlet problems $\{m_\varepsilon\}_{\varepsilon\in]0,1]}$, $m_\varepsilon : W^{1,p}(\Omega;\mathbb{R}^m) \times \mathcal{O}(\Omega) \to [0,\infty]$ defined by
$$m_\varepsilon(u;O) := \inf\left\{F_\varepsilon(v;O) : W^{1,p}(\Omega;\mathbb{R}^m) \ni v = u \text{ in } \Omega\backslash O\right\}.$$
Note that we can write
$$m_\varepsilon(u;O) = \inf\left\{F_\varepsilon(v;O) : v \in u + W^{1,p}_0(O;\mathbb{R}^m)\right\}$$
since $u + W^{1,p}_0(O;\mathbb{R}^m) = \left\{v \in W^{1,p}(\Omega;\mathbb{R}^m) : v - u = 0 \text{ in } \Omega\backslash O\right\}$ (see [AH96, p. 234, Theorem 9.1.3]).

Remark 2.1. The functional $m_\varepsilon(\cdot;O)$ can be seen as the "quotient functional" $\widetilde{F_\varepsilon}(\cdot;O)$ defined on the quotient space of $W^{1,p}(\Omega;\mathbb{R}^m)$ by $W^{1,p}_0(O;\mathbb{R}^m)$, i.e.,
$$\widetilde{F_\varepsilon}(\cdot;O) : {}^{W^{1,p}(\Omega;\mathbb{R}^m)}\!/\!_{W^{1,p}_0(O;\mathbb{R}^m)} \to [0,\infty]$$
$$\text{with } \widetilde{F_\varepsilon}([u];O) := \inf_{v\in[u]} F_\varepsilon(v;O) = m_\varepsilon(u;O)$$
where $[u] = u + W^{1,p}_0(O;\mathbb{R}^m)$ is the equivalent class of u.

2.2. **A general $\Gamma(L^p)$-convergence theorem.** We denote by λ the Lebesgue measure on Ω. For each $O \in \mathcal{O}(\Omega)$ we denote by $\mathfrak{A}_\lambda(O)$ the space of nonnegative *finite* Borel measures on O which are absolutely continuous with respect to the Lebesgue measure $\lambda\lfloor_O$ on O.

Let us introduce the M-*sets associated to* $\mathcal{F}$: for each $O \in \mathcal{O}(\Omega)$ we set
$$\mathrm{M}_{\mathcal{F}}(O) := \left\{u \in W^{1,p}(\Omega;\mathbb{R}^m) : \exists \mu_u \in \mathfrak{A}_\lambda(O) \quad \sup_{\varepsilon>0} m_\varepsilon(u;\cdot) \leqslant \mu_u(\cdot) \text{ on } O\right\}. \tag{2.1}$$

- We assume that all the affine maps, i.e., functions of the form $u(x) = v + \zeta x$ with $(x, v, \zeta) \in \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times d}$, belong to $\mathrm{M}_{\mathcal{F}}(O)$.

We will see in Theorem 2.2 that the M-set is the set where an integral representation of the $\Gamma(L^p)$-limit is possible.

To the family $\mathcal{F}$ we associate $m_+ : W^{1,p}(\Omega;\mathbb{R}^m) \times \mathcal{O}(\Omega) \to [0,\infty]$ defined by

$$m_+(u;O) := \overline{\lim_{\varepsilon\to 0}}\, m_\varepsilon(u;O).$$

The following result provides bounds in integral forms of both $\Gamma(L^p)$-$\underline{\lim}_{\varepsilon\to 0}$ and $\Gamma(L^p)$-$\overline{\lim}_{\varepsilon\to 0}$ of a family $\mathcal{F} = \{F_\varepsilon\}_{\varepsilon\in]0,1]} \subset \mathcal{I}(p,\alpha)$, i.e., satisfying ($\mathrm{C}_1$), ($\mathrm{C}_2$) and ($\mathrm{C}_3$).

Theorem 2.1. *Let $\mathcal{F} = \{F_\varepsilon\}_{\varepsilon\in]0,1]} \subset \mathcal{I}(p,\alpha)$ and let $(u,O) \in W^{1,p}(\Omega;\mathbb{R}^m) \times \mathcal{O}(\Omega)$.*

(i) If $u \in \mathrm{M}_{\mathcal{F}}(O)$ then

$$\mathcal{F}_+(u;O) \leqslant \mathcal{F}_+^{\mathfrak{D}}(u;O) \leqslant \int_O \lim_{\rho\to 0} \inf\left\{\frac{m_+(u;\mathrm{Q})}{\lambda(\mathrm{Q})} : x \in \mathrm{Q} \in \mathcal{Q}_o(O),\ \mathrm{diam}(\mathrm{Q}) \leqslant \rho\right\} dx$$

where

$$\mathcal{F}_+^{\mathfrak{D}}(u;O) := \inf\left\{\overline{\lim_{\varepsilon\to 0}}\, F_\varepsilon(v_\varepsilon;O) : W_0^{1,p}(O;\mathbb{R}^m) + u \ni v_\varepsilon \to u \text{ in } L^p(\Omega;\mathbb{R}^m)\right\};$$

(ii) There exists $\{u_{\varepsilon_n}\}_n \subset W^{1,p}(\Omega;\mathbb{R}^m)$ with $\sup_n F_{\varepsilon_n}(u_{\varepsilon_n};O) < \infty$ such that $u_{\varepsilon_n} \to u$ in $L^p(\Omega;\mathbb{R}^m)$ as $n \to \infty$ and

$$\mathcal{F}_-^{\mathfrak{D}}(u;O) \geqslant \mathcal{F}_-(u;O) \geqslant \int_O \lim_{\rho\to 0} \lim_{n\to\infty} \frac{F_{\varepsilon_n}(u_{\varepsilon_n};\mathrm{Q}_\rho(x))}{\rho^d} dx$$

where

$$\mathcal{F}_-^{\mathfrak{D}}(u;O) := \inf\left\{\underline{\lim_{\varepsilon\to 0}}\, F_\varepsilon(v_\varepsilon;O) : W_0^{1,p}(O;\mathbb{R}^m) + u \ni v_\varepsilon \to u \text{ in } L^p(\Omega;\mathbb{R}^m)\right\}.$$

Remark 2.2. If $u \in \mathrm{M}_{\mathcal{F}}(O)$ then there exists $\mu_u \in \mathfrak{A}_\lambda(O)$ such that $\sup_{\varepsilon>0} m_\varepsilon(u;\cdot) \leqslant \mu_u(\cdot)$ on O. Therefore we have

$$m_+(u;\cdot) \leqslant \mu_u(\cdot) \text{ on } O.$$

Taking account of Theorem 2.1 (i) we deduce that $\mathcal{F}_+(u;O) < \infty$, which means that

$$\mathrm{M}_{\mathcal{F}}(O) \subset \mathrm{dom}\mathcal{F}_+^{\mathfrak{D}}(\cdot;O) := \left\{u \in W^{1,p}(\Omega;\mathbb{R}^m) : \mathcal{F}_+^{\mathfrak{D}}(u;O) < \infty\right\} \subset \mathrm{dom}\mathcal{F}_+(\cdot;O).$$

To the family $\mathcal{F} = \{F_\varepsilon\}_{\varepsilon\in]0,1]}$ we associate $m_- : W^{1,p}(\Omega;\mathbb{R}^m) \times \mathcal{O}(\Omega) \to [0,\infty]$ defined by

$$m_-(u;O) := \underline{\lim_{\varepsilon\to 0}}\, m_\varepsilon(u;O).$$

Let $O \in \mathcal{O}(\Omega)$ and let $u \in W^{1,p}(O;\mathbb{R}^m)$. We denote the affine tangent map of u at $x \in O$ by

$$u_x(\cdot) := u(x) + \nabla u(x)(\cdot - x).$$

Consider the following local inequalities for $u \in \mathrm{M}_{\mathcal{F}}(O)$:

(H_1) $\displaystyle \overline{\lim_{\rho\to 0}} \frac{m_-(u_x;\mathrm{Q}_\rho(x))}{\rho^d} \geqslant \overline{\lim_{\rho\to 0}} \frac{m_+(u_x;\mathrm{Q}_\rho(x))}{\rho^d}$ a.e. in O;

(H_2) $\displaystyle \overline{\lim_{\rho\to 0}}\, \underline{\lim_{\varepsilon\to 0}} \frac{F_\varepsilon(u_\varepsilon;\mathrm{Q}_\rho(x))}{\rho^d} \geqslant \overline{\lim_{\rho\to 0}} \frac{m_-(u_x;\mathrm{Q}_\rho(x))}{\rho^d}$ a.e. in O for all $\{u_\varepsilon\}_\varepsilon \subset W^{1,p}(\Omega;\mathbb{R}^m)$ such that $u_\varepsilon \to u$ in $L^p(\Omega;\mathbb{R}^m)$ and $\sup_\varepsilon F_\varepsilon(u_\varepsilon;O) < \infty$;

(H_3) $\displaystyle \underline{\lim_{\rho\to 0}} \frac{m_+(u_x;\mathrm{Q}_\rho(x))}{\rho^d} \geqslant \underline{\lim_{\rho\to 0}} \frac{m_+(u;\mathrm{Q}_\rho(x))}{\rho^d}$ a.e. in O.

Remark 2.3. We make some remarks on the previous inequalities.

(*i*) Similar condition to (H_1), related to the integral representation of the $\Gamma(L^p)$-limit of functionals of the calculus of variations, is already known when p-polynomial growth (and convexity conditions) is assumed see [Mas06, p. 451]. In fact, since $m_+(u;O) \geqslant m_-(u;O)$ for all $(u,O) \in W^{1,p}(\Omega;\mathbb{R}^m) \times \mathcal{O}(\Omega)$, the condition (H_1) can be rewritten as follows

$$\varlimsup_{\rho\to 0} \frac{m_-(u_x;\mathrm{Q}_\rho(x))}{\rho^d} = \varlimsup_{\rho\to 0} \frac{m_+(u_x;\mathrm{Q}_\rho(x))}{\rho^d} \quad \text{a.e. in } O.$$

We see that a sufficient condition for (H_1) to hold is

$$\lim_{\varepsilon\to 0} \frac{m_\varepsilon(u_x;\mathrm{Q})}{\lambda(\mathrm{Q})} \quad \text{exists for all open cube } \mathrm{Q} \subset O \text{ and all } x \in O.$$

(*ii*) The condition (H_2) (resp. (H_3)) can be seen as a "local" $\Gamma(L^p)$-$\varliminf$ (resp. $\Gamma(L^p)$-$\varlimsup$) inequality. To verify inequality (H_3) (resp. (H_2)) we need to replace u (resp. a sequence $\{u_\varepsilon\}_\varepsilon$ converging in L^p to u and satisfying $\sup_\varepsilon F_\varepsilon(u_\varepsilon;O) < \infty$) by the affine tangent map u_x in the localization of m_ε on "small" cubes $\mathrm{Q}_\rho(x)$. This can be performed, for instance, by using growth conditions see Sect. 5.

(*iii*) If $\{F_\varepsilon = F\}_\varepsilon = \mathcal{F}$ is constant with respect to ε then (H_1) is always satisfied.

The following lemma is used in the proof of Theorem 2.2 and its proof is given in Sect. 4.

Lemma 2.1. *Let $\mathcal{F} = \{F_\varepsilon\}_{\varepsilon\in]0,1]} \subset \mathcal{I}(p,\alpha)$. Let $O \in \mathcal{O}(\Omega)$ and let $u \in \mathrm{M}_\mathcal{F}(O)$. If (H_1), (H_2) and (H_3) hold then the following limit $\lim_{\rho\to 0} \frac{m_+(u_x;\mathrm{Q}_\rho(x))}{\rho^d}$ exists almost everywhere in O. Moreover the function $O \ni x \mapsto \lim_{\rho\to 0} \frac{m_+(u_x;\mathrm{Q}_\rho(x))}{\rho^d}$ is measurable and satisfies*

$$\lim_{\rho\to 0} \frac{m_+(u_x;\mathrm{Q}_\rho(x))}{\rho^d} = \varlimsup_{\rho\to 0} \frac{m_-(u_x;\mathrm{Q}_\rho(x))}{\rho^d} = \varliminf_{\rho\to 0} \frac{m_+(u;\mathrm{Q}_\rho(x))}{\rho^d} \quad \textit{a.e. in } O.$$

Here is the general $\Gamma(L^p)$-convergence theorem which shows that under the local inequalities (H_1), (H_2) and (H_3) the family $\mathcal{F}(\cdot;O)$ $\Gamma(L^p)$-converges to an integral functional of the calculus of variations at every $u \in \mathrm{M}_\mathcal{F}(O)$. In Sect. 5 we give applications to homogenization and relaxation of this result. When $\mathcal{F}_+^{\mathfrak{D}} = \mathcal{F}_-^{\mathfrak{D}}$ we denote by $\mathcal{F}_0^{\mathfrak{D}} = \mathcal{F}_+^{\mathfrak{D}} = \mathcal{F}_-^{\mathfrak{D}}$ the common value.

Theorem 2.2. *Let $\mathcal{F} = \{F_\varepsilon\}_{\varepsilon\in]0,1]} \subset \mathcal{I}(p,\alpha)$. Let $O \in \mathcal{O}(\Omega)$ and let $u \in \mathrm{M}_\mathcal{F}(O)$. If (H_1), (H_2) and (H_3) hold then the family of functionals $\mathcal{F}(\cdot;O)$ $\Gamma(L^p)$-converges at u to*

$$\mathcal{F}_0(u;O) = \mathcal{F}_0^{\mathfrak{D}}(u;O) = \int_O \lim_{\rho\to 0} \frac{m_+(u_x;\mathrm{Q}_\rho(x))}{\rho^d} dx. \tag{2.2}$$

Moreover, we have for almost all $x \in O$

$$\lim_{\rho\to 0} \frac{m_+(u_x;\mathrm{Q}_\rho(x))}{\rho^d} = \varlimsup_{\rho\to 0} \frac{m_-(u_x;\mathrm{Q}_\rho(x))}{\rho^d}. \tag{2.3}$$

Proof. Let $O \in \mathcal{O}(\Omega)$ and let $u \in \mathrm{M}_\mathcal{F}(O)$. From Theorem 2.1 (*ii*), there exists $\{u_\varepsilon\}_\varepsilon \subset W^{1,p}(\Omega;\mathbb{R}^m)$ with $\sup_\varepsilon F_\varepsilon(u_\varepsilon;O) < \infty$ such that $u_\varepsilon \to u$ in $L^p(\Omega;\mathbb{R}^m)$ as $\varepsilon \to 0$ and

$$\mathcal{F}_-^{\mathfrak{D}}(u;O) \geqslant \mathcal{F}_-(u;O) \geqslant \int_O \varlimsup_{\rho\to 0} \varliminf_{\varepsilon\to 0} \frac{F_\varepsilon(u_\varepsilon;\mathrm{Q}_\rho(x))}{\rho^d} dx$$

Using the local inequalities (H_1), (H_2) and (H_3), Lemma 2.1 together with Theorem 2.1 we have

$$\begin{aligned}
\mathcal{F}_+(u;O) \leqslant \mathcal{F}_+^{\mathfrak{D}}(u;O) &\leqslant \int_O \lim_{\rho\to 0} \inf\left\{\frac{m_+(u;\mathrm{Q})}{\lambda(\mathrm{Q})} : x \in \mathrm{Q} \in \mathcal{Q}_o(O)\,,\ \mathrm{diam}(\mathrm{Q}) \leqslant \rho\right\} dx \\
&\leqslant \int_O \varliminf_{\rho\to 0} \frac{m_+(u;\mathrm{Q}_\rho(x))}{\rho^d} dx \\
&\leqslant \int_O \lim_{\rho\to 0} \frac{m_+(u_x;\mathrm{Q}_\rho(x))}{\rho^d} dx \\
&= \int_O \varlimsup_{\rho\to 0} \frac{m_-(u_x;\mathrm{Q}_\rho(x))}{\rho^d} dx \\
&\leqslant \int_O \varlimsup_{\rho\to 0} \varliminf_{\varepsilon\to 0} \frac{F_\varepsilon(u_\varepsilon;\mathrm{Q}_\rho(x))}{\rho^d} dx \\
&\leqslant \mathcal{F}_-(u;O) \leqslant \mathcal{F}_-^{\mathfrak{D}}(u;O).
\end{aligned}$$

Thus (2.2) holds. The equality (2.3) is a consequence of Lemma 2.1. ■

2.3. **On the condition ($\mathbf{H_1}$).** The following result shows that the condition (H_1) holds whenever $\mathcal{F}_-^{\mathfrak{D}}$ and $\mathcal{F}_+^{\mathfrak{D}}$ are equal.

Proposition 2.1. *If for every cube* $\mathrm{Q} \in \mathcal{O}(\Omega)$ *and every* $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ *we have for every* $x \in \Omega$

$$\inf\left\{\mathcal{F}_-^{\mathfrak{D}}(v;\mathrm{Q}) : v \in u_x + W_0^{1,p}(\mathrm{Q};\mathbb{R}^m)\right\} = \inf\left\{\mathcal{F}_+^{\mathfrak{D}}(v;\mathrm{Q}) : v \in u_x + W_0^{1,p}(\mathrm{Q};\mathbb{R}^m)\right\}$$

then (H_1) *holds.*

Proof. Let $\mathrm{Q} \in \mathcal{O}(\Omega)$ be a cube, $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ and $x \in \Omega$. We set

$$l(u_x;\mathrm{Q}) := \inf\left\{\mathcal{F}_-^{\mathfrak{D}}(v;\mathrm{Q}) : v \in u_x + W_0^{1,p}(\mathrm{Q};\mathbb{R}^m)\right\} = \inf\left\{\mathcal{F}_+^{\mathfrak{D}}(v;\mathrm{Q}) : v \in u_x + W_0^{1,p}(\mathrm{Q};\mathbb{R}^m)\right\}.$$

Let $\{\varepsilon_n\}_n \subset]0,1]$ be a sequence such that $\lim_{n\to\infty}\varepsilon_n = 0$. There exists a sequence $\{v_n\}_n \subset u_x + W_0^{1,p}(\mathrm{Q};\mathbb{R}^m)$ such that

$$\varliminf_{n\to\infty} F_{\varepsilon_n}(v_n;\mathrm{Q}) = \varliminf_{n\to\infty} m_{\varepsilon_n}(u_x;\mathrm{Q}) \leqslant \mu_{u_x}(\mathrm{Q}) < \infty.$$

since $u_x \in \mathrm{M}_{\mathcal{F}}(\mathrm{Q})$. There exists a subsequence such that

$$\varliminf_{n\to\infty} m_{\varepsilon_n}(u;\mathrm{Q}) = \varliminf_{n\to\infty} F_{\varepsilon_{\sigma(n)}}\left(v_{\sigma(n)};\mathrm{Q}\right) \text{ and } v_{\sigma(n)} \to v_\infty \in u + W_0^{1,p}(\mathrm{Q};\mathbb{R}^m) \text{ in } L^p(\Omega;\mathbb{R}^m)$$

since p-coercivity. Therefore

$$\varliminf_{n\to\infty} m_{\varepsilon_n}(u_x;\mathrm{Q}) \geqslant \mathcal{F}_-^{\mathfrak{D}}(v_\infty;\mathrm{Q}) \geqslant l(u_x;\mathrm{Q}).$$

For any $v \in u_x + W_0^{1,p}(\mathrm{Q};\mathbb{R}^m)$ we have

$$\varlimsup_{n\to\infty} m_{\varepsilon_n}(u_x;\mathrm{Q}) \leqslant \mathcal{F}_+^{\mathfrak{D}}(v;\mathrm{Q})$$

that means

$$\varlimsup_{n\to\infty} m_{\varepsilon_n}(u_x;\mathrm{Q}) \leqslant l(u_x;\mathrm{Q}).$$

We deduce that the following limit exists for every cube $\mathrm{Q} \in \mathcal{O}(\Omega)$ and every $u \in W^{1,p}(\Omega;\mathbb{R}^m)$

$$l(u_x;\mathrm{Q}) = \lim_{n\to\infty} m_{\varepsilon_n}(u_x;\mathrm{Q}) = \varliminf_{\varepsilon\to 0} m_\varepsilon(u_x;\mathrm{Q}) = \varlimsup_{\varepsilon\to 0} m_\varepsilon(u_x;\mathrm{Q}).$$

It follows that

$$\varlimsup_{\rho\to 0}\lim_{n\to\infty}\frac{m_{\varepsilon_n}(u_x;\mathrm{Q}_\rho(x))}{\rho^d} = \varlimsup_{\rho\to 0}\varliminf_{\varepsilon\to 0}\frac{m_\varepsilon(u_x;\mathrm{Q}_\rho(x))}{\rho^d} = \varlimsup_{\rho\to 0}\varlimsup_{\varepsilon\to 0}\frac{m_\varepsilon(u_x;\mathrm{Q}_\rho(x))}{\rho^d}. \quad ■$$

Remark 2.4. Let $O \in \mathcal{O}(\Omega)$ and $u \in \mathrm{M}_{\mathcal{F}}(O)$. Assume that ($\mathrm{H}_2$) and ($\mathrm{H}_3$) hold. Then ($\mathrm{H}_1$) holds if and only if $\mathcal{F}_+^{\mathfrak{D}}(u;O) = \mathcal{F}_-^{\mathfrak{D}}(u;O)$.

2.4. **The relaxation case.** We examine the particular case of a constant family with respect to the parameter $\mathcal{F} = \{F_\varepsilon = F\}_\varepsilon \subset \mathcal{I}(p,\alpha)$. We set for every $(u,O) \in W^{1,p}(\Omega;\mathbb{R}^m) \times \mathcal{O}(\Omega)$

$$\begin{aligned}
\mathcal{F}_0(u;O) &:= \inf\left\{\varliminf_{\varepsilon\to 0} F(v_\varepsilon;O) : W^{1,p}(\Omega;\mathbb{R}^m) \ni v_\varepsilon \to u \text{ in } L^p(\Omega;\mathbb{R}^m)\right\};\\
\mathcal{F}_0^{\mathfrak{D}}(u;O) &:= \inf\left\{\varliminf_{\varepsilon\to 0} F(v_\varepsilon;O) : W_0^{1,p}(O;\mathbb{R}^m) + u \ni v_\varepsilon \to u \text{ in } L^p(\Omega;\mathbb{R}^m)\right\};\\
m(u;O) &:= \inf\left\{F(v;O) : v \in u + W_0^{1,p}(O;\mathbb{R}^m)\right\}.
\end{aligned}$$

The following abstract relaxation result is a direct consequence of Remarks 2.3 (*iii*) and Theorem 2.2.

Proposition 2.2. *Let* $O \in \mathcal{O}(\Omega)$ *and let* $u \in \mathrm{M}_{\mathcal{F}}(O)$. *If* ($\mathrm{H}_2$) *and* ($\mathrm{H}_3$) *hold then*

$$\mathcal{F}_0(u;O) = \mathcal{F}_0^{\mathfrak{D}}(u;O) = \int_O \lim_{\rho\to 0}\frac{m(u_x;\mathrm{Q}_\rho(x))}{\rho^d}dx. \tag{2.4}$$

Remark 2.5. In particular (2.4) holds for all $u \in \mathrm{dom}F(\cdot;O)$ since it is easy to see that $\mathrm{dom}F(\cdot;O) \subset \mathrm{M}_{\mathcal{F}}(O)$.

2.5. **Remarks on the limit integrand.** We assume that the assumptions of Theorem 2.2 hold. We give descriptions of the limit integrand L_0 by considering some particular cases.

(*i*) If we define $\widetilde{L}_0 : \Omega \times \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0,\infty]$ by

$$\widetilde{L}_0\left(x,v,\xi\right) := \overline{\lim_{\rho\to 0}} \frac{m_+\left(v+\xi\left(\cdot - x\right);\mathrm{Q}_\rho\left(x\right)\right)}{\rho^d}, \tag{2.5}$$

and for each $u \in \mathrm{M}_{\mathcal{F}}\left(O\right)$

$$L_0\left(x,u\left(x\right),\nabla u\left(x\right)\right) := \lim_{\rho\to 0} \frac{m_+\left(u_x;\mathrm{Q}_\rho\left(x\right)\right)}{\rho^d} \tag{2.6}$$

then the formula (2.2) becomes

$$\mathcal{F}_0\left(u;O\right) = \int_O \widetilde{L}_0\left(x,u\left(x\right),\nabla u\left(x\right)\right)dx.$$

Indeed, we have for every $x \in O$

$$L_0\left(x,u\left(x\right),\nabla u\left(x\right)\right) = \widetilde{L}_0\left(x,u\left(x\right),\nabla u\left(x\right)\right). \tag{2.7}$$

In fact, we do not know whether the integrand $\widetilde{L}_0$ is Borel measurable. Because of the equality (2.7), the function $O \ni x \mapsto \widetilde{L}_0\left(x,u\left(x\right),\nabla u\left(x\right)\right)$ is measurable for all $u \in \mathrm{M}_{\mathcal{F}}\left(O\right)$.

(*ii*) Assume that $\{F_\varepsilon\}_\varepsilon = \mathcal{F}$ is given under integral form, i.e., $F_\varepsilon : W^{1,p}\left(\Omega;\mathbb{R}^m\right) \times \mathcal{O}\left(\Omega\right) \to [0,\infty]$ is defined by

$$F_\varepsilon\left(u;O\right) := \int_O L_\varepsilon\left(x,u\left(x\right),\nabla u\left(x\right)\right)dx$$

where $L_\varepsilon : \Omega \times \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0,\infty]$ is Borel measurable for all $\varepsilon \in]0,1]$. Then

$$\begin{aligned}&\widetilde{L}_0\left(x,v,\xi\right) = \\ &\overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}} \inf\left\{\fint_{\mathrm{Q}_\rho(x)} L_\varepsilon\left(y,v+\xi\left(y-x\right)+\varphi\left(y\right),\xi+\nabla\varphi\left(y\right)\right)dy : \varphi \in W_0^{1,p}\left(\mathrm{Q}_\rho\left(x\right);\mathbb{R}^m\right)\right\}.\end{aligned} \tag{2.8}$$

If, moreover, we assume that L_ε does not depend of the variable v, i.e., $L_\varepsilon : \Omega \times \mathbb{M}^{m\times d} \to [0,\infty]$ then (2.7) becomes

$$L_0\left(x,\nabla u\left(x\right)\right) = \widetilde{L}_0\left(x,\nabla u\left(x\right)\right)$$

for all $x \in O$ and all $u \in \mathrm{M}_{\mathcal{F}}\left(O\right)$. Since the affine functions belong to $\mathrm{M}_{\mathcal{F}}\left(O\right)$ we deduce that for every $x \in O$ and every $\xi \in \mathbb{M}^{m\times d}$

$$L_0\left(x,\xi\right) = \widetilde{L}_0\left(x,\xi\right).$$

(*iii*) Now, we consider the case where $\{F_\varepsilon = F\}_\varepsilon = \mathcal{F}$ is constant with respect to ε and $F : W^{1,p}\left(\Omega;\mathbb{R}^m\right) \times \mathcal{O}\left(\Omega\right) \to [0,\infty]$ is defined by

$$F\left(u;O\right) := \int_O L\left(x,u\left(x\right),\nabla u\left(x\right)\right)dx$$

where $L : \Omega \times \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0,\infty]$ is Borel measurable. If we define for every $(x,v,\xi) \in \Omega \times \mathbb{R}^d \times \mathbb{M}^{m\times d}$

$$\begin{aligned}&\widetilde{L}_0\left(x,v,\xi\right) = \\ &\overline{\lim_{\rho\to 0}} \inf\left\{\fint_{\mathrm{Q}_\rho(x)} L\left(y,v+\xi\left(y-x\right)+\varphi\left(y\right),\xi+\nabla\varphi\left(y\right)\right)dy : \varphi \in W_0^{1,p}\left(\mathrm{Q}_\rho\left(x\right);\mathbb{R}^m\right)\right\}.\end{aligned} \tag{2.9}$$

then for every $u \in \mathrm{M}_{\mathcal{F}}\left(O\right)$ and every $x \in O$

$$L_0\left(x,u\left(x\right),\nabla u\left(x\right)\right) = \widetilde{L}_0\left(x,u\left(x\right),\nabla u\left(x\right)\right).$$

We will show in Proposition 5.1 that when L is Carathéodory with p-growth and p-coercivity we recover the classical quasiconvex envelope [Dac08, Theorem 9.8, p. 432].

3. Integral representation of Vitali envelope and derivation of set functions

3.1. **Integral representation of Vitali envelope of set functions.** For a given open set $O \subset \Omega$ we denote by $\mathcal{Q}_o(O)$ the set of all open cube of O. We denote by $\mathcal{Q}_c(O)$ the set of all closed cube of O.

Let $G : \mathcal{Q}_o(\Omega) \to]-\infty, \infty]$ be a set function. We define the *Vitali envelope* of G with respect to λ

$$\mathcal{O}(\Omega) \ni O \mapsto V_G(O) := \sup_{\varepsilon>0} \inf \left\{ \sum_{i\in I} G(\mathrm{Q}_i) : \{\overline{\mathrm{Q}}_i\}_{i\in I} \in \mathcal{V}^\varepsilon(O) \right\}$$

where for any $\varepsilon > 0$

$$\mathcal{V}^\varepsilon(O) := \Big\{ \{\overline{\mathrm{Q}}_i\}_{i\in I} \subset \mathcal{Q}_c(\Omega) : I \text{ is countable, } \lambda\Big(O \backslash \underset{i\in I}{\cup} \mathrm{Q}_i\Big) = 0,\ \ \overline{\mathrm{Q}}_i \subset O$$

$$\operatorname{diam}(\mathrm{Q}_i) \in]0, \varepsilon[\ \text{ and } \ \overline{\mathrm{Q}}_i \cap \overline{\mathrm{Q}}_j = \emptyset \ \text{ for all } \ i \neq j \Big\}.$$

Remark 3.1. If G is the trace on $\mathcal{Q}_o(\Omega)$ of a positive Borel measure ν on Ω which is absolutely continuous with respect to λ then $V_G(O) = \nu(O)$ for all $O \in \mathcal{O}(\Omega)$.

Let $G : \mathcal{Q}_o(\Omega) \to]-\infty, \infty]$ be a set function. Define the upper and the lower derivatives at $x \in \Omega$ of G with respect to λ as follows

$$\underline{D}_\lambda G(x) := \lim_{\rho\to 0} \inf \left\{ \frac{G(\mathrm{Q})}{\lambda(\mathrm{Q})} : x \in \mathrm{Q} \in \mathcal{Q}_o(\Omega),\ \operatorname{diam}(\mathrm{Q}) \leqslant \rho \right\};$$

$$\overline{D}_\lambda G(x) := \lim_{\rho\to 0} \sup \left\{ \frac{G(\mathrm{Q})}{\lambda(\mathrm{Q})} : x \in \mathrm{Q} \in \mathcal{Q}_o(\Omega),\ \operatorname{diam}(\mathrm{Q}) \leqslant \rho \right\}.$$

We say that G is λ-differentiable in O if for λ-almost all $x \in O$ it holds

$$-\infty < \underline{D}_\lambda G(x) = \overline{D}_\lambda G(x) < \infty.$$

Remark 3.2. For every $O \in \mathcal{O}(\Omega)$ and every $x \in O$ we have

$$\underline{D}_\lambda G(x) := \lim_{\rho\to 0} \inf \left\{ \frac{G(\mathrm{Q})}{\lambda(\mathrm{Q})} : x \in \mathrm{Q} \in \mathcal{Q}_o(O),\ \operatorname{diam}(\mathrm{Q}) \leqslant \rho \right\};$$

$$\overline{D}_\lambda G(x) := \lim_{\rho\to 0} \sup \left\{ \frac{G(\mathrm{Q})}{\lambda(\mathrm{Q})} : x \in \mathrm{Q} \in \mathcal{Q}_o(O),\ \operatorname{diam}(\mathrm{Q}) \leqslant \rho \right\}.$$

The proof of the following classical result can be found in Appendix.

Lemma 3.1. *The functions $\underline{D}_\lambda G(\cdot)$ and $\overline{D}_\lambda G(\cdot)$ are λ-measurable.*

Remark 3.3. When $G = \nu$ is a Borel finite measure absolutely continuous with respect to λ then ν is λ-differentiable in O and

$$D_\lambda \nu(x) = \lim_{\rho\to 0} \frac{\nu(\mathrm{Q}_\rho(x))}{\rho^d} \qquad \text{a.e. in } O.$$

The following result establishes an integral representation for the Vitali envelope of nonnegative set functions.

Proposition 3.1. *Let $H : \mathcal{Q}_o(\Omega) \to [0, \infty]$ be a set function. For every $O \in \mathcal{O}(\Omega)$ we have*

$$V_H(O) = \int_O \underline{D}_\lambda H(y)\, dy.$$

The following lemma was inspired by reading [Bon82].

Lemma 3.2. *Let $G : \mathcal{Q}_o(\Omega) \to]-\infty, \infty]$ be a set function and $O \in \mathcal{O}(\Omega)$.*

(*a*) *If $\underline{D}_\lambda G(x) \leqslant 0$ λ-a.e. in O then $V_G(O) \leqslant 0$.*
(*b*) *If $\underline{D}_\lambda G(x) \geqslant 0$ λ-a.e. in O then $V_G(O) \geqslant 0$.*
(*c*) *If $\underline{D}_\lambda G(x) = 0$ λ-a.e. in O then $V_G(O) = 0$.*

Proof. The assertion (c) is a consequence of (a) and (b).

Proof of (a). It is enough to show that for every $\varepsilon > 0$ if

$$\underline{D}_{\lambda} G(x) < \varepsilon \quad \lambda\text{-a.e. in } O \tag{3.1}$$

then

$$\inf\left\{\sum_{i\in I} G(\mathrm{Q}_i) : \{\overline{\mathrm{Q}}_i\}_{i\in I} \in \mathcal{V}_{\varepsilon}(O)\right\} < \varepsilon\lambda(O).$$

Fix $\varepsilon > 0$. Let $N \subset O$ with $\lambda(N) = 0$ be such that $O\backslash N = [\underline{D}_{\lambda} G(\cdot) < \varepsilon]$. Using Lemma 6.1 (i) with $h = \eta = \varepsilon$ and $S_h = O\backslash N$, there exists a countable pairwise disjointed family $\{\mathrm{Q}_i\}_{i\in I} \subset \mathcal{Q}_o(O)$ such that

$$\lambda\left((O\backslash N)\backslash \underset{i\in I}{\cup} \mathrm{Q}_i\right) = 0, \quad \forall i \in I \quad G(\mathrm{Q}_i) < \varepsilon\lambda(\mathrm{Q}_i) \text{ and } \operatorname{diam}(\mathrm{Q}_i) < \varepsilon. \tag{3.2}$$

From (3.2) we have $\lambda\left(O\backslash \cup_{i\in I} \overline{\mathrm{Q}}_i\right) = 0$ since $\lambda(N) = 0$. Consequently, we have $\sum_{i\in I} \lambda\left(\overline{\mathrm{Q}}_i\right) = \sum_{i\in I} \lambda(\mathrm{Q}_i) = \lambda(O)$. Summing over $i \in I$ the first inequality (3.2) we obtain

$$\inf\left\{\sum_{i\in I} G(\mathrm{Q}_i) : \{\overline{\mathrm{Q}}_i\}_{i\in I} \in \mathcal{V}_{\varepsilon}(O)\right\} < \varepsilon \sum_{i\in I} \lambda(\mathrm{Q}_i) = \varepsilon\lambda(O).$$

Proof of (b). Let $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset]0,1[$ be such that $\lim_{n\to\infty} \varepsilon_n = 0$. Let $n \in \mathbb{N}$. There exists $\{\mathrm{Q}_i^n\}_{i\in I_n} \in \mathcal{V}_{\varepsilon_n}(O)$ such that

$$\inf\left\{\sum_{i\in I} G(\mathrm{Q}_i) : \{\mathrm{Q}_i\} \in \mathcal{V}_{\varepsilon_n}(O)\right\} + \frac{1}{n} \geqslant \sum_{i\in I_n} \frac{G(\mathrm{Q}_i^n)}{\lambda(\mathrm{Q}_i^n)} \lambda(\mathrm{Q}_i^n). \tag{3.3}$$

Fix $x \in O$ be such that

$$\underline{D}_{\lambda} G(x) \geqslant 0 \quad \text{and} \quad x \notin \bigcup_{n\in\mathbb{N}} \left(O\backslash \underset{i\in I_n}{\cup} \mathrm{Q}_i^n\right).$$

There exists $i_x^n \in I_n$ such that $x \in \mathrm{Q}_{i_x^n}^n$. From (3.3) it follows that

$$\begin{aligned}
&\inf\left\{\sum_{i\in I} G(\mathrm{Q}_i) : \{\mathrm{Q}_i\} \in \mathcal{V}_{\varepsilon_n}(O)\right\} + \frac{1}{n} \\
&\geqslant \frac{G\left(\mathrm{Q}_{i_x^n}^n\right)}{\lambda\left(\mathrm{Q}_{i_x^n}^n\right)} \lambda\left(\mathrm{Q}_{i_x^n}^n\right) \\
&\geqslant \inf\left\{\frac{G(\mathrm{Q})}{\lambda(\mathrm{Q})} : x \in \mathrm{Q} \in \mathcal{Q}_o(O),\ \operatorname{diam}(\mathrm{Q}) \leqslant \varepsilon_n\right\} \lambda\left(\mathrm{Q}_{i_x^n}^n\right).
\end{aligned} \tag{3.4}$$

Passing to the limit $n \to \infty$ in (3.4) we obtain

$$V_G(O) \geqslant \underline{D}_{\lambda} G(x) \varliminf_{n\to\infty} \lambda\left(\mathrm{Q}_{i_x^n}^n\right) \geqslant 0.$$

The proof is complete. ■

Corollary 3.1. *If $G = H - \nu$ where $H : \mathcal{Q}_o(\Omega) \to]-\infty, \infty]$ is a set function, ν is a finite Borel measure on $O \in \mathcal{O}(\Omega)$ absolutely continuous with respect to $\lambda\lfloor_O$, and if*

$$\underline{D}_{\lambda} G(x) = 0 \quad \lambda\text{-a.e. in } O$$

then

$$V_H(O) = \nu(O).$$

Proof. Use Lemma 3.2(c) and remark that $V_G(O) = V_H(O) - \nu(O)$. ■

Proof of Proposition 3.1. Assume that $\underline{D}_\lambda H \in L^1(O)$, i.e.,

$$\int_O \underline{D}_\lambda H(y)\,dy < \infty.$$

Define $G : \mathcal{Q}_o(O) \to]-\infty,\infty]$ by

$$G(\mathrm{Q}) := H(\mathrm{Q}) - \int_{\mathrm{Q}} \underline{D}_\lambda H(y)\,dy.$$

If we show that $\underline{D}_\lambda G = 0$ a.e. in O then by Corollary 3.1 we can conclude that

$$V_H(O) = \nu(O) \text{ with } \nu := \underline{D}_\lambda H(\cdot)\,\lambda\lfloor_O.$$

Fix $x \in O$ such that

$$\underline{D}_\lambda H(x) = \lim_{\rho\to 0} \fint_{\mathrm{Q}_\rho(x)} \underline{D}_\lambda H(y)\,dy < \infty; \tag{3.5}$$

$$\underline{D}_\lambda \nu(x) = \underline{D}_\lambda H(x) < \infty. \tag{3.6}$$

We set $\mathfrak{B}_{x,\rho} := \{\mathrm{Q} : x \in \mathrm{Q} \in \mathcal{Q}_o(O) \text{ and } \operatorname{diam}(\mathrm{Q}) \leqslant \rho\}$ for all $\rho > 0$. On one hand, we have for every $\rho > 0$ and every $\mathrm{Q} \in \mathfrak{B}_{x,\rho}$

$$\begin{aligned}\frac{G(\mathrm{Q})}{\lambda(\mathrm{Q})} &\leqslant \frac{H(\mathrm{Q})}{\lambda(\mathrm{Q})} + \sup\left\{-\fint_{\mathrm{Q}} \underline{D}_\lambda H(y)\,dy : \mathrm{Q} \in \mathfrak{B}_{x,\rho}\right\}\\ &= \frac{H(\mathrm{Q})}{\lambda(\mathrm{Q})} - \inf\left\{\fint_{\mathrm{Q}} \underline{D}_\lambda H(y)\,dy : \mathrm{Q} \in \mathfrak{B}_{x,\rho}\right\}.\end{aligned}$$

Taking the infimum over every $\mathrm{Q} \in \mathfrak{B}_{x,\rho}$ we obtain

$$\begin{aligned}&\inf\left\{\frac{G(\mathrm{Q})}{\lambda(\mathrm{Q})} : \mathrm{Q} \in \mathfrak{B}_{x,\rho}\right\}\\ &\leqslant \inf\left\{\frac{H(\mathrm{Q})}{\lambda(\mathrm{Q})} : \mathrm{Q} \in \mathfrak{B}_{x,\rho}\right\} - \inf\left\{\fint_{\mathrm{Q}} \underline{D}_\lambda H(y)\,dy : \mathrm{Q} \in \mathfrak{B}_{x,\rho}\right\}.\end{aligned}$$

Letting $\rho \to 0$ and using (3.5) and (3.6), we have

$$\underline{D}_\lambda G(x) \leqslant \underline{D}_\lambda H(x) - \underline{D}_\lambda H(x) = 0. \tag{3.7}$$

On the other hand, we have for every $\rho > 0$ and every $\mathrm{Q} \in \mathfrak{B}_{x,\rho}$

$$\begin{aligned}\frac{G(\mathrm{Q})}{\lambda(\mathrm{Q})} &\geqslant \frac{H(\mathrm{Q})}{\lambda(\mathrm{Q})} + \inf\left\{-\fint_{\mathrm{Q}} \underline{D}_\lambda H(y)\,dy : \mathrm{Q} \in \mathfrak{B}_{x,\rho}\right\}\\ &= \frac{H(\mathrm{Q})}{\lambda(\mathrm{Q})} - \sup\left\{\fint_{\mathrm{Q}} \underline{D}_\lambda H(y)\,dy : \mathrm{Q} \in \mathfrak{B}_{x,\rho}\right\}.\end{aligned}$$

Taking the infimum over every $\mathrm{Q} \in \mathfrak{B}_{x,\rho}$ we obtain

$$\begin{aligned}&\inf\left\{\frac{G(\mathrm{Q})}{\lambda(\mathrm{Q})} : \mathrm{Q} \in \mathfrak{B}_{x,\rho}\right\}\\ &\leqslant \inf\left\{\frac{H(\mathrm{Q})}{\lambda(\mathrm{Q})} : \mathrm{Q} \in \mathfrak{B}_{x,\rho}\right\} - \sup\left\{\fint_{\mathrm{Q}} \underline{D}_\lambda H(y)\,dy : \mathrm{Q} \in \mathfrak{B}_{x,\rho}\right\}.\end{aligned}$$

Letting $\rho \to 0$ and using (3.5) and (3.6), we have

$$\underline{D}_\lambda G(x) \geqslant \underline{D}_\lambda H(x) - \underline{D}_\lambda H(x) = 0. \tag{3.8}$$

Taking account of (3.7) and (3.8), we finally obtain that $\underline{D}_\lambda G(x) = 0$.

Now, we do not assume that $\underline{D}_\lambda H \in L^1(O)$, in this case the following inequality is always true

$$V_H(O) \leqslant \int_O \underline{D}_\lambda H(y)\,dy.$$

It remains to prove the opposite inequality. For every $n \in \mathbb{N}$ we set $H_n : \mathcal{Q}_0(\Omega) \to [0,\infty[$ defined by

$$H_n(\mathrm{Q}) := \begin{cases} H(\mathrm{Q}) & \text{if } H(\mathrm{Q}) \leqslant n\lambda(\mathrm{Q})\\[2ex] n\lambda(\mathrm{Q}) & \text{if } H(\mathrm{Q}) > n\lambda(\mathrm{Q}).\end{cases}$$

It is easy to see that

$$
\begin{aligned}
(3.9)\qquad & \forall \mathrm{Q} \in \mathcal{Q}_0(\Omega) && H_0(\mathrm{Q}) \leqslant H_1(\mathrm{Q}) \leqslant \cdots \leqslant H_n(\mathrm{Q}) \leqslant \cdots \leqslant \sup_{n\in\mathbb{N}} H_n(\mathrm{Q}) \leqslant H(\mathrm{Q});\\
& \forall n \in \mathbb{N} && \underline{D}_\lambda H_n \leqslant n.
\end{aligned}
$$

So $\{\underline{D}_\lambda H_n\}_{n\in\mathbb{N}} \subset L^1(O)$, we apply the first part of the proof to have

$$
\forall n \in \mathbb{N} \qquad V_{H_n}(O) = \int_O \underline{D}_\lambda H_n(y)\,dy \leqslant V_H(O).
$$

Using (3.9) and monotone convergence theorem we have

$$
\sup_{n\in\mathbb{N}} V_{H_n}(O) = \int_O \sup_{n\in\mathbb{N}} \underline{D}_\lambda H_n(y)\,dy \leqslant V_H(O). \tag{3.10}
$$

Fix $n \in \mathbb{N}$ and $x \in [\underline{D}_\lambda H \leqslant n]$. Then for every $\rho > 0$ we have

$$
\inf_{\mathrm{Q}\in\mathfrak{B}_{x,\rho}} \frac{H(\mathrm{Q})}{\lambda(\mathrm{Q})} \leqslant n. \tag{3.11}
$$

For each $n \in \mathbb{N}$ and each $\rho > 0$ we set $A_n := \{\mathrm{Q} \in \mathfrak{B}_{x,\rho} : H(\mathrm{Q}) \leqslant n\lambda(\mathrm{Q})\}$ and $B_n := \mathfrak{B}_{x,\rho} \backslash A_n$. Then

$$
\begin{aligned}
\underline{D}_\lambda H_n(x) &= \sup_{\rho>0} \inf_{\mathrm{Q}\in\mathfrak{B}_{x,\rho}} \frac{H_n(\mathrm{Q})}{\lambda(\mathrm{Q})}\\
&= \sup_{\rho>0} \min\left\{ \inf_{\mathrm{Q}\in A_n} \frac{H_n(\mathrm{Q})}{\lambda(\mathrm{Q})}, \inf_{\mathrm{Q}\in B_n} \frac{H_n(\mathrm{Q})}{\lambda(\mathrm{Q})} \right\}\\
&= \sup_{\rho>0} \min\left\{ \inf_{\mathrm{Q}\in A_n} \frac{H(\mathrm{Q})}{\lambda(\mathrm{Q})}, n \right\}\\
&\geqslant \sup_{\rho>0} \min\left\{ \inf_{\mathrm{Q}\in\mathfrak{B}_{x,\rho}} \frac{H(\mathrm{Q})}{\lambda(\mathrm{Q})}, n \right\}.
\end{aligned}
$$

Using (3.11) we find

$$
\underline{D}_\lambda H_n(x) \geqslant \sup_{\rho>0} \inf_{\mathrm{Q}\in\mathfrak{B}_{x,\rho}} \frac{H(\mathrm{Q})}{\lambda(\mathrm{Q})} = \underline{D}_\lambda H(x).
$$

It follows that $\sup_{n\in\mathbb{N}} \underline{D}_\lambda H_n(x) = \underline{D}_\lambda H(x)$ for all $x \in O$ and thus (3.10) becomes

$$
\int_O \underline{D}_\lambda H(y)\,dy \leqslant V_H(O).
$$

The proof is complete. ■

4. Proof of main results

4.1. **Proof of Lemma 2.1.** Fix $O \in \mathcal{O}(\Omega)$ and $u \in \mathrm{M}_{\mathcal{F}}(O)$. By Theorem 2.1 *(ii)* there exists $\{u_{\varepsilon_n}\}_{\varepsilon_n}$ with $\sup_n F_{\varepsilon_n}(u_{\varepsilon_n};\Omega) < \infty$ such that $u_{\varepsilon_n} \to u$ in $L^p(\Omega;\mathbb{R}^m)$ as $n \to \infty$ and

$$
\infty > \mathcal{F}_-^{\mathfrak{D}}(u;O) \geqslant \mathcal{F}_-(u;O) \geqslant \int_O \varliminf_{\rho\to 0} \varliminf_{n\to\infty} \frac{F_{\varepsilon_n}(u_{\varepsilon_n};\mathrm{Q}_\rho(x))}{\rho^d} dx.
$$

since Remark 2.2. It follows that for almost all $x \in O$

$$
\varlimsup_{\rho\to 0} \frac{\mathcal{F}_-(u;\mathrm{Q}_\rho(x))}{\rho^d} \geqslant \varliminf_{\rho\to 0} \varliminf_{n\to\infty} \frac{F_{\varepsilon_n}(u_{\varepsilon_n};\mathrm{Q}_\rho(x))}{\rho^d}. \tag{4.1}
$$

Using the local inequalities (H_1), (H_2), (H_3), (4.1) and Theorem 2.1 (*i*) we have for almost all $x \in O$

$$
\begin{aligned}
\varlimsup_{\rho \to 0} \frac{\mathcal{F}_+\left(u; \mathrm{Q}_\rho(x)\right)}{\rho^d} &\leqslant \varlimsup_{\rho \to 0} \frac{\mathcal{F}_+^{\mathfrak{D}}\left(u; \mathrm{Q}_\rho(x)\right)}{\rho^d} \\
&\leqslant \underline{D}_\lambda m_+\left(u; \cdot\right)(x) \\
&\leqslant \varliminf_{\rho \to 0} \frac{m_+\left(u; \mathrm{Q}_\rho(x)\right)}{\rho^d} \\
&\leqslant \varliminf_{\rho \to 0} \frac{m_+\left(u_x; \mathrm{Q}_\rho(x)\right)}{\rho^d} \\
&\leqslant \varlimsup_{\rho \to 0} \frac{m_+\left(u_x; \mathrm{Q}_\rho(x)\right)}{\rho^d} \\
&\leqslant \varlimsup_{\rho \to 0} \frac{m_-\left(u_x; \mathrm{Q}_\rho(x)\right)}{\rho^d} \\
&\leqslant \varliminf_{\rho \to 0} \lim_{n \to \infty} \frac{F_{\varepsilon_n}\left(u_{\varepsilon_n}; \mathrm{Q}_\rho(x)\right)}{\rho^d} \\
&\leqslant \varliminf_{\rho \to 0} \frac{\mathcal{F}_-\left(u; \mathrm{Q}_\rho(x)\right)}{\rho^d}. \qquad (4.2)
\end{aligned}
$$

From the last inequality (4.2) we have the following inequalities

$$
\varliminf_{\rho \to 0} \frac{\mathcal{F}_-\left(u; \mathrm{Q}_\rho(x)\right)}{\rho^d} \leqslant \varliminf_{\rho \to 0} \frac{\mathcal{F}_+\left(u; \mathrm{Q}_\rho(x)\right)}{\rho^d} \leqslant \varlimsup_{\rho \to 0} \frac{\mathcal{F}_+\left(u; \mathrm{Q}_\rho(x)\right)}{\rho^d}
$$

and

$$
\varliminf_{\rho \to 0} \frac{\mathcal{F}_-\left(u; \mathrm{Q}_\rho(x)\right)}{\rho^d} \leqslant \varlimsup_{\rho \to 0} \frac{\mathcal{F}_-\left(u; \mathrm{Q}_\rho(x)\right)}{\rho^d} \leqslant \varlimsup_{\rho \to 0} \frac{\mathcal{F}_+\left(u; \mathrm{Q}_\rho(x)\right)}{\rho^d}
$$

and

$$
\varliminf_{\rho \to 0} \frac{\mathcal{F}_-\left(u; \mathrm{Q}_\rho(x)\right)}{\rho^d} \leqslant \varliminf_{\rho \to 0} \frac{\mathcal{F}_+^{\mathfrak{D}}\left(u; \mathrm{Q}_\rho(x)\right)}{\rho^d} \leqslant \varlimsup_{\rho \to 0} \frac{\mathcal{F}_+^{\mathfrak{D}}\left(u; \mathrm{Q}_\rho(x)\right)}{\rho^d}
$$

and

$$
\begin{aligned}
\varliminf_{\rho \to 0} \frac{\mathcal{F}_-\left(u; \mathrm{Q}_\rho(x)\right)}{\rho^d} &\leqslant \varliminf_{\rho \to 0} \frac{\mathcal{F}_-^{\mathfrak{D}}\left(u; \mathrm{Q}_\rho(x)\right)}{\rho^d} \leqslant \varlimsup_{\rho \to 0} \frac{\mathcal{F}_-^{\mathfrak{D}}\left(u; \mathrm{Q}_\rho(x)\right)}{\rho^d} \\
&\leqslant \varlimsup_{\rho \to 0} \frac{\mathcal{F}_+^{\mathfrak{D}}\left(u; \mathrm{Q}_\rho(x)\right)}{\rho^d}
\end{aligned}
$$

for all $x \in O$. It follows that for almost all $x \in O$

$$
\begin{aligned}
\lim_{\rho \to 0} \frac{\mathcal{F}_-\left(u; \mathrm{Q}_\rho(x)\right)}{\rho^d} &= \lim_{\rho \to 0} \frac{\mathcal{F}_+\left(u; \mathrm{Q}_\rho(x)\right)}{\rho^d} \\
&= \lim_{\rho \to 0} \frac{\mathcal{F}_+^{\mathfrak{D}}\left(u; \mathrm{Q}_\rho(x)\right)}{\rho^d} = \lim_{\rho \to 0} \frac{\mathcal{F}_-^{\mathfrak{D}}\left(u; \mathrm{Q}_\rho(x)\right)}{\rho^d} \\
&= \underline{D}_\lambda m_+\left(u; \cdot\right)(x) = \varliminf_{\rho \to 0} \frac{m_+\left(u; \mathrm{Q}_\rho(x)\right)}{\rho^d} \\
&= \lim_{\rho \to 0} \frac{m_+\left(u_x; \mathrm{Q}_\rho(x)\right)}{\rho^d} = \varlimsup_{\rho \to 0} \frac{m_-\left(u_x; \mathrm{Q}_\rho(x)\right)}{\rho^d}.
\end{aligned}
$$

So, the proof is complete since $O \ni x \mapsto \underline{D}_\lambda m_+\left(u; \cdot\right)(x)$ is measurable by Lemma 3.1. ■

4.2. **Proof of Theorem 2.1.**

Proof of Theorem 2.1 (ii). Let $(u, O) \in W^{1,p}\left(\Omega; \mathbb{R}^m\right) \times \mathcal{O}(\Omega)$ be such that $\mathcal{F}_-(u; O) < \infty$. There exists a sequence $\{u_{\varepsilon_n}\}_n \subset W^{1,p}\left(\Omega; \mathbb{R}^m\right)$ such that

$$
u_{\varepsilon_n} \to u \text{ in } L^p\left(\Omega; \mathbb{R}^m\right), \quad \lim_{n \to \infty} F_{\varepsilon_n}\left(u_{\varepsilon_n}; O\right) = \mathcal{F}_-(u; O) \text{ and } \sup_n F_{\varepsilon_n}\left(u_{\varepsilon_n}; O\right) < \infty. \qquad (4.3)
$$

By (C_2), for each $\varepsilon > 0$ we consider the Borel measure ν_ε whose the trace on $\mathcal{O}(\Omega)$ is $F_\varepsilon\left(u_\varepsilon; \cdot\right)$. From the last inequality of (4.3) we can rewrite that the sequence of Borel measures $\{\mu_n := \nu_{\varepsilon_n}\lfloor_O\}_n$ satisfies $\sup_n \mu_n(O) < \infty$. So, there exists a Borel measure μ on O such that (up to a subsequence) $\mu_n \overset{*}{\rightharpoonup} \mu$.

By Lebesgue decomposition theorem, we have $\mu = \mu_a + \mu_s$ where μ_a and μ_s are nonnegative Borel measures such that $\mu_a \ll \lambda\lfloor_O$ and $\mu_s \perp \lambda\lfloor_O$, and from Radon-Nikodym theorem we deduce that there exists $f \in L^1(O;\mathbb{R}^+)$, given by

$$f(x) = \lim_{\rho\to 0} \frac{\mu_a(\mathrm{Q}_\rho(x))}{\rho^d} = \lim_{\rho\to 0} \frac{\mu(\mathrm{Q}_\rho(x))}{\rho^d} \quad \text{a.e. in } O$$

with $\mathrm{Q}_\rho(x) := x + \rho Y$, such that

$$\mu_a(A) = \int_A f(x)\,dx \text{ for all measurable sets } A \subset O.$$

By Alexandrov theorem we see that

$$\begin{aligned}\mathcal{F}_-(u;O) &= \lim_{n\to\infty} F_{\varepsilon_n}(u_{\varepsilon_n};O)\\ &= \lim_{n\to\infty} \mu_n(O) \geqslant \mu(O) = \mu_a(O) + \mu_s(O) \geqslant \mu_a(O) = \int_O f(x)\,dx,\end{aligned}$$

and

$$f(x) = \lim_{\rho\to 0}\lim_{n\to\infty} \frac{\mu_n(\mathrm{Q}_\rho(x))}{\rho^d} = \lim_{\rho\to 0}\lim_{n\to\infty} \frac{F_{\varepsilon_n}(u_{\varepsilon_n};\mathrm{Q}_\rho(x))}{\rho^d} \quad \text{a.e. in } O. \quad \blacksquare$$

Proof of Theorem 2.1 (i). For each $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ we denote by $V_+(u;\cdot) : \mathcal{O}(\Omega) \to [0,\infty]$ the Vitali envelope of $m_+(u;\cdot)$, i.e.,

$$V_+(u;O) := V_{m_+(u;\cdot)}(O).$$

The proof consists to show that for every $O \in \mathcal{O}(\Omega)$ and every $u \in \mathrm{M}_{\mathcal{F}}(O)$ the following inequality holds

$$\mathcal{F}_+^{\mathfrak{D}}(u;O) \leqslant V_+(u;O). \tag{4.4}$$

Indeed, using Proposition 3.1 we obtain

$$\mathcal{F}_+(u;O) \leqslant \mathcal{F}_+^{\mathfrak{D}}(u;O) \leqslant V_+(u;O) = \int_O \underline{D}_\lambda m_+(u;\cdot)(x)\,dx.$$

Let us prove (4.4) now. Fix $O \in \mathcal{O}(\Omega)$ and $u \in \mathrm{M}_{\mathcal{F}}(O)$. Note that by Remarks 3.1 we have for some $\mu_u \in \mathfrak{A}_\lambda(O)$

$$V_+(u;O) \leqslant \mu_u(O) < \infty. \tag{4.5}$$

Fix $\varepsilon \in]0,1[$. Choose $\{\overline{\mathrm{Q}}_i\}_{i\in I} \in \mathcal{V}_\varepsilon(O)$ such that

$$\sum_{i\in I} m_+(u;\mathrm{Q}_i) \leqslant V_+^\varepsilon(u;O) + \frac{\varepsilon}{2} \leqslant V_+(u;O) + \frac{\varepsilon}{2}. \tag{4.6}$$

Fix $\delta \in]0,1[$. Given any $i \in I$, by definition of $m_\delta(u;\mathrm{Q}_i)$, there exists $v_i \in u + W_0^{1,p}(\mathrm{Q}_i;\mathbb{R}^m)$ such that

$$F_\delta(v_i;\mathrm{Q}_i) \leqslant m_\delta(u;\mathrm{Q}_i) + \frac{\delta}{2}\frac{\lambda(\mathrm{Q}_i)}{\lambda(O)}. \tag{4.7}$$

Define $u_{\delta,\varepsilon} \in u + W_0^{1,p}(O;\mathbb{R}^m)$ by

$$u_{\delta,\varepsilon} := \sum_{i\in I} v_i \mathbb{1}_{\mathrm{Q}_i} + u\mathbb{1}_{\Omega\setminus \underset{i\in I}{\cup} \mathrm{Q}_i}.$$

Using (C_2) and (C_3) we have from (4.7)

$$\begin{aligned}F_\delta(u_{\delta,\varepsilon};O) &= \sum_{i\in I} F_\delta(v_i;\mathrm{Q}_i) + F_\delta\left(u;O\setminus \underset{i\in I}{\cup}\mathrm{Q}_i\right)\\ &= \sum_{i\in I} F_\delta(v_i;\mathrm{Q}_i)\\ &\leqslant \sum_{i\in I} m_\delta(u;\mathrm{Q}_i) + \frac{\delta}{2}.\end{aligned}$$

Since $u \in \mathrm{M}_{\mathcal{F}}(O)$ there exists $\mu_u \in \mathfrak{A}_\lambda(O)$ such that $\sup_{\delta\in]0,1]} m_\delta(u;U) \leqslant \mu_u(U)$ for all open set $U \subset O$. For every $\eta > 0$ there exists a finite set $I_\eta \subset I$ such that $\mu_u\left(O \setminus \cup_{i\in I_\eta} \mathrm{Q}_i\right) \leqslant \eta$. It follows that $\sum_{i\in I\setminus I_\eta} m_\delta(u;\mathrm{Q}_i) \leqslant \eta$. Hence, for any $\eta > 0$

$$\overline{\lim_{\delta\to 0}} \sum_{i\in I} m_\delta(u;\mathrm{Q}_i) \leqslant \overline{\lim_{\delta\to 0}} \sum_{i\in I_\eta} m_\delta(u;\mathrm{Q}_i) + \overline{\lim_{\delta\to 0}} \sum_{i\in I\setminus I_\eta} m_\delta(u;\mathrm{Q}_i) \tag{4.8}$$
$$\leqslant \sum_{i\in I} m_+(u;\mathrm{Q}_i) + \eta.$$

Therefore collecting (4.6), (4.8), and passing to the limit $\varepsilon \to 0$, we have

$$\overline{\lim_{\varepsilon\to 0}}\,\overline{\lim_{\delta\to 0}} F_\delta(u_{\delta,\varepsilon};O) \leqslant V_+(u;O). \tag{4.9}$$

From the p-coercivity (C_1), (4.9) and (4.5), we deduce

$$\overline{\lim_{\varepsilon\to 0}}\,\overline{\lim_{\delta\to 0}} \int_O |\nabla u_{\delta,\varepsilon}|^p dx < \infty. \tag{4.10}$$

By Poincaré inequality there exists $K > 0$ depending only on p and d such that for each $v_i \in u + W_0^{1,p}(\mathrm{Q}_i;\mathbb{R}^m)$

$$\int_{\mathrm{Q}_i} |v_i - u|^p dx \leqslant K\varepsilon^p \int_{\mathrm{Q}_i} |\nabla v_i - \nabla u|^p dx$$

since $\mathrm{diam}(\mathrm{Q}_i) < \varepsilon$. Summing over $i \in I$ we obtain

$$\int_O |u_{\delta,\varepsilon} - u|^p dx \leqslant 2^{p-1} K\varepsilon^p \left(\int_O |\nabla u_{\delta,\varepsilon}|^p dx + \int_O |\nabla u|^p dx\right)$$

which shows, by using (4.10), that

$$\overline{\lim_{\varepsilon\to 0}}\,\overline{\lim_{\delta\to 0}} \int_\Omega |u_{\delta,\varepsilon} - u|^p dx = 0. \tag{4.11}$$

A simultaneous diagonalization of (4.9) and (4.11) gives a sequence $\{u_\delta := u_{\delta,\varepsilon(\delta)}\}_\delta \subset u + W^{1,p}(O;\mathbb{R}^m)$ such that $u_\delta \to u$ in $L^p(\Omega;\mathbb{R}^m)$ and

$$\mathcal{F}_+^{\mathfrak{D}}(u;O) \leqslant \overline{\lim_{\delta\to 0}} F_\delta(u_\delta;O) \leqslant V_+(u;O)$$

by the definition of $\mathcal{F}_+^{\mathfrak{D}}(u;O)$. The proof is complete. ■

5. Applications

5.1. General $\Gamma(L^p)$-convergence result in the p-growth case. For each $\varepsilon \in]0,1]$ we consider a family of functionals $\mathcal{F} := \{F_\varepsilon\}_{\varepsilon\in]0,1]}$, $F_\varepsilon : W^{1,p}(\Omega;\mathbb{R}^m) \times \mathcal{O}(\Omega) \to [0,\infty]$.

Consider the following condition:

(P_1) there exist $\beta > 0$ and ν a nonnegative finite Borel measure on Ω absolutely continuous with respect to the Lebesgue measure such that for every $(V,u,\varepsilon) \in \mathcal{O}(\Omega) \times W^{1,p}(\Omega;\mathbb{R}^m) \times]0,1]$ we have

$$\frac{m_\varepsilon(u;V)}{|V|} \leqslant \beta\left(\frac{\nu(V)}{|V|} + \fint_V |u|^p dx + \fint_V |\nabla u|^p dx\right)$$

The following result can be seen as a nonconvex extension of Theorem IV of [DMM86a, p. 265]. Indeed, if for each $\varepsilon > 0$ we set $F_\varepsilon : W^{1,p}(\Omega;\mathbb{R}^m) \times \mathcal{O}(\Omega) \to [0,\infty]$ defined by

$$F_\varepsilon(u;O) := \int_O L_\varepsilon(x, u(x), \nabla u(x))\, dx$$

where $L_\varepsilon : \Omega \times \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0,\infty[$ is a Borel measurable function with p-growth and p-coercivity, i.e.,

$$\exists \alpha > 0 \quad \exists \beta > 0 \quad \exists a \in L^1(\Omega) \quad \forall (x,v,\xi) \in \Omega \times \mathbb{R}^d \times \mathbb{M}^{m\times d} \quad \forall \varepsilon > 0 \tag{5.1}$$

$$\alpha|\xi|^p \leqslant L_\varepsilon(x,v,\xi) \leqslant \beta\left(a(x) + |v|^p + |\xi|^p\right)$$

then (P_1) holds with $\nu = a(\cdot)\lambda$ and $\mathcal{F} = \{F_\varepsilon\}_\varepsilon \subset \mathcal{I}(p,\alpha)$.

Theorem 5.1. *Assume that $\mathcal{F} \subset \mathcal{I}(p,\alpha)$. Let $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ and $O \in \mathcal{O}(\Omega)$. If* (H$_1$) *and* (P$_1$) *hold then the family $\mathcal{F}(\cdot;O)$ $\Gamma(L^p)$-converges at u to*

$$\mathcal{F}_0(u,O) := \int_O L_0(x,u(x),\nabla u(x))\,dx$$

where $L_0(\cdot,u(\cdot),\nabla u(\cdot))$ is given by (2.6).

Proof. Since (P$_1$) we see that

$$\mathrm{M}_{\mathcal{F}}(O) = W^{1,p}(\Omega;\mathbb{R}^m).$$

Fix $u \in W^{1,p}(\Omega;\mathbb{R}^m)$. Following Theorem 2.2 it is enough to show that (H$_2$) and (H$_3$) hold.

We begin by showing (H$_3$). Fix $x \in O$ such that

$$\lim_{r\to 0} \fint_{\mathrm{Q}_r(x)} |u|^p dy = |u(x)|^p < \infty; \tag{5.2}$$

$$\lim_{r\to 0} \fint_{\mathrm{Q}_r(x)} |\nabla u|^p dy = |\nabla u(x)|^p < \infty; \tag{5.3}$$

$$\lim_{r\to 0} \frac{1}{r^p} \fint_{\mathrm{Q}_r(x)} |u_x - u|^p dy = 0; \tag{5.4}$$

$$\lim_{r\to 0} \frac{\nu(\mathrm{Q}_r(x))}{r^d} = D_\lambda \nu(x) < \infty. \tag{5.5}$$

Fix $\varepsilon > 0$, $s \in]0,1[$ and $\rho > 0$. Let $\phi \in W_0^{1,\infty}(\mathrm{Q}_\rho(x);[0,1])$ be a cut-off function between $\overline{\mathrm{Q}}_{s\rho}(x)$ and $\mathrm{Q}_\rho(x)$ (i.e., $\phi = 1$ on $\overline{\mathrm{Q}}_{s\rho}(x)$ and $\phi = 0$ on $O \backslash \mathrm{Q}_{s\rho}(x)$) such that

$$\|\nabla \phi\|_{L^\infty(\mathrm{Q}_\rho(x))} \leqslant \frac{4}{\rho(1-s)}.$$

Let $v_\varepsilon \in u_x + W_0^{1,p}(\mathrm{Q}_{s\rho}(x);\mathbb{R}^m)$ be such that

$$F_\varepsilon(v_\varepsilon;\mathrm{Q}_{s\rho}(x)) \leqslant \varepsilon (s\rho)^d + m_\varepsilon(u_x;\mathrm{Q}_{s\rho}(x)). \tag{5.6}$$

Set $w := \phi v_\varepsilon + (1-\phi)u$, we have $w \in u + W_0^{1,p}(\mathrm{Q}_\rho(x);\mathbb{R}^m)$ and

$$\nabla w := \begin{cases} \nabla v_\varepsilon & \text{in } \mathrm{Q}_{s\rho}(x) \\ \phi \nabla u(x) + (1-\phi)\nabla u + \nabla \phi \otimes (u_x - u) & \text{in } \Sigma_\rho(x) \end{cases}$$

where $\Sigma_\rho(x) := \mathrm{Q}_\rho(x) \backslash \overline{\mathrm{Q}}_{s\rho}(x)$. We have

$$\begin{aligned} m_\varepsilon(u;\mathrm{Q}_\rho(x)) &= m_\varepsilon(w;\mathrm{Q}_\rho(x)) \\ &= m_\varepsilon(w;\mathrm{Q}_{s\rho}(x)) + m_\varepsilon(w;\Sigma_\rho(x)) \\ &\leqslant F_\varepsilon(v_\varepsilon;\mathrm{Q}_{s\rho}(x)) + m_\varepsilon(w;\Sigma_\rho(x)) \\ &\leqslant \varepsilon (s\rho)^d + m_\varepsilon(u_x;\mathrm{Q}_{s\rho}(x)) + m_\varepsilon(w;\Sigma_\rho(x)) \end{aligned}$$

since Lemma 6.2 and (5.6). It follows that

$$\frac{m_\varepsilon(u;\mathrm{Q}_\rho(x))}{\rho^d} \leqslant \varepsilon s^d + s^d \frac{m_\varepsilon(u_x;\mathrm{Q}_{s\rho}(x))}{(s\rho)^d} + \frac{m_\varepsilon(w;\Sigma_\rho(x))}{\rho^d}. \tag{5.7}$$

We claim that (H$_3$) is proved if

$$\overline{\lim_{s\to 1}}\, \overline{\lim_{\rho\to 0}}\, \overline{\lim_{\varepsilon\to 0}} \frac{m_\varepsilon(w;\Sigma_\rho(x))}{\rho^d} = 0. \tag{5.8}$$

Indeed, passing to the limits $\varepsilon \to 0$, $\rho \to 0$, $s \to 1$ in (5.7) we have

$$\begin{aligned} \lim_{\rho\to 0} \frac{m_+(u;\mathrm{Q}_\rho(x))}{\rho^d} &\leqslant \lim_{s\to 1^-} \lim_{\rho\to 0} \frac{m_+(u_x;\mathrm{Q}_{s\rho}(x))}{(s\rho)^d} \\ &\leqslant \lim_{\rho\to 0} \frac{m_+(u_x;\mathrm{Q}_\rho(x))}{\rho^d}. \end{aligned} \tag{5.9}$$

To verify the last inequality in (5.9), we write by using Lemma 6.2

$$\begin{aligned} m_\varepsilon\left(u_x;\mathrm{Q}_\rho\left(x\right)\right) &= m_\varepsilon\left(u_x;\mathrm{Q}_{s\rho}\left(x\right)\right)+m_\varepsilon\left(u_x;\Sigma_\rho\left(x\right)\right) \\ &\geqslant m_\varepsilon\left(u_x;\mathrm{Q}_{s\rho}\left(x\right)\right), \end{aligned}$$

dividing by ρ^d and taking the $\overline{\lim}$ as $\varepsilon\to 0$ we have

$$\frac{m_+\left(u_x;\mathrm{Q}_\rho\left(x\right)\right)}{\rho^d}\geqslant s^d\frac{m_+\left(u_x;\mathrm{Q}_{s\rho}\left(x\right)\right)}{\left(s\rho\right)^d}.$$

Passing to the limits $\rho\to 0$ and then $s\to 1$ we obtain the last inequality in (5.9).

So, it remains to prove (5.8). Using (P_1) we have for some $C>0$ dependent on p only

$$\begin{aligned}
&\frac{m_\varepsilon\left(w;\Sigma_\rho\left(x\right)\right)}{\rho^d}\\
&\leqslant \beta\left(\frac{\nu\left(\Sigma_\rho\left(x\right)\right)}{\rho^d}+\frac{1}{\rho^d}\int_{\Sigma_\rho(x)}\left|\phi\nabla u\left(x\right)+\left(1-\phi\right)\nabla u+\nabla\phi\otimes\left(u_x-u\right)\right|^p dy\right)\\
&+\frac{\beta}{\rho^d}\int_{\Sigma_\rho(x)}\left|\phi u_x+\left(1-\phi\right)u\right|^p dy\\
&\leqslant C\beta\left(\frac{\nu\left(\Sigma_\rho\left(x\right)\right)}{\rho^d}+\left(1-s^d\right)\left|\nabla u\left(x\right)\right|^p+\fint_{\mathrm{Q}_\rho(x)}\left|\nabla u\right|^p dy-s^d\fint_{\mathrm{Q}_{s\rho}(x)}\left|\nabla u\right|^p dy\right)\\
&+C\beta\left(\frac{4^p}{\left(1-s\right)^p}\left(\frac{1}{\rho^p}\fint_{\mathrm{Q}_\rho(x)}\left|u_x-u\right|^p dy-\frac{s^{d+p}}{\left(s\rho\right)^p}\fint_{\mathrm{Q}_{s\rho}(x)}\left|u_x-u\right|^p dy\right)\right)\\
&+C\beta\rho^p\left(\frac{1}{\rho^p}\fint_{\mathrm{Q}_\rho(x)}\left|u_x-u\right|^p dy-\frac{s^{d+p}}{\left(s\rho\right)^p}\fint_{\mathrm{Q}_{s\rho}(x)}\left|u_x-u\right|^p dy\right)\\
&+C\beta\left(\fint_{\mathrm{Q}_\rho(x)}\left|u\right|^p-s^d\fint_{\mathrm{Q}_{s\rho}(x)}\left|u\right|^p\right).
\end{aligned}$$

Taking (5.2), (5.3), (5.4) and (5.5) into account and passing to the limits $\varepsilon\to 0$ then $\rho\to 0$ we obtain

$$\overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\frac{m_\varepsilon\left(w;\Sigma_\rho\left(x\right)\right)}{\rho^d}\leqslant C\beta\left(1-s^d\right)\left(D_\lambda\nu\left(x\right)+\left|u\left(x\right)\right|^p+\left|\nabla u\left(x\right)\right|^p\right).$$

Letting $s\to 1$ we obtain (5.8).

Let us prove (H_2) now. Consider a sequence $\{\varphi_\varepsilon\}_\varepsilon\subset W^{1,p}\left(\Omega;\mathbb{R}^m\right)$ such that $\varphi_\varepsilon\to 0$ in $L^p\left(\Omega;\mathbb{R}^m\right)$ as $\varepsilon\to 0$ and satisfying $\sup_{\varepsilon>0}F_\varepsilon\left(u+\varphi_\varepsilon;\Omega\right)<\infty$. Set $\mu_\varepsilon\left(\cdot\right):=F_\varepsilon\left(u+\varphi_\varepsilon;\cdot\right)$ for any $\varepsilon>0$. There exists a subsequence (not relabeled) and a nonnegative Radon measure μ_0 such that

$$\mu_\varepsilon\overset{*}{\rightharpoonup}\mu_0. \tag{5.10}$$

Fix $\varepsilon>0, s\in]1,2[$ and $\rho>0$. Fix $x\in O$ such that (5.2), (5.3), (5.4) and (5.5) hold and

$$D_\lambda\mu_0\left(x\right):=\lim_{r\to 0}\frac{\mu_0\left(\mathrm{Q}_r\left(x\right)\right)}{r^d}<\infty. \tag{5.11}$$

Let $\phi\in W_0^{1,\infty}\left(\mathrm{Q}_{s\rho}\left(x\right);[0,1]\right)$ be a cut-off function between $\overline{\mathrm{Q}}_\rho\left(x\right)$ and $\mathrm{Q}_{s\rho}\left(x\right)$ such that

$$\left\|\nabla\phi\right\|_{L^\infty(\mathrm{Q}_{s\rho}(x))}\leqslant\frac{4}{\rho\left(s-1\right)}.$$

Let $v_\varepsilon\in\left(u+\varphi_\varepsilon\right)+W_0^{1,p}\left(\mathrm{Q}_\rho\left(x\right);\mathbb{R}^m\right)$ be such that

$$F_\varepsilon\left(v_\varepsilon;\mathrm{Q}_\rho\left(x\right)\right)\leqslant\varepsilon\rho^d+m_\varepsilon\left(u+\varphi_\varepsilon;\mathrm{Q}_\rho\left(x\right)\right). \tag{5.12}$$

Set $w:=\phi v_\varepsilon+\left(1-\phi\right)u_x$, we have $w\in u_x+W_0^{1,p}\left(\mathrm{Q}_{s\rho}\left(x\right);\mathbb{R}^m\right)$ and

$$\nabla w:=\begin{cases}\nabla v_\varepsilon & \text{in } \mathrm{Q}_\rho\left(x\right)\\ \phi\left(\nabla u+\nabla\varphi_\varepsilon\right)+\left(1-\phi\right)\nabla u\left(x\right)+\nabla\phi\otimes\left(u+\varphi_\varepsilon-u_x\right) & \text{in } \Sigma_\rho\left(x\right)\end{cases}$$

where $\Sigma_\rho(x) := Q_{s\rho}(x) \backslash \overline{Q}_\rho(x)$. We have

$$
\begin{aligned}
s^d \frac{m_\varepsilon(u_x; Q_{s\rho}(x))}{(s\rho)^d} &= s^d \frac{m_\varepsilon(w; Q_{s\rho}(x))}{(s\rho)^d} \\
&= \frac{m_\varepsilon(w; Q_\rho(x))}{\rho^d} + \frac{m_\varepsilon(w; \Sigma_\rho(x))}{\rho^d} \\
&\leqslant \frac{F_\varepsilon(v_\varepsilon; Q_\rho(x))}{\rho^d} + \frac{m_\varepsilon(w; \Sigma_\rho(x))}{\rho^d} \\
&\leqslant \varepsilon + \frac{m_\varepsilon(u+\varphi_\varepsilon; Q_\rho(x))}{\rho^d} + \frac{m_\varepsilon(w; \Sigma_\rho(x))}{\rho^d} \\
&\leqslant \varepsilon + \frac{F_\varepsilon(u+\varphi_\varepsilon; Q_\rho(x))}{\rho^d} + \frac{m_\varepsilon(w; \Sigma_\rho(x))}{\rho^d}
\end{aligned} \tag{5.13}
$$

since Lemma 6.2 and (5.12). We claim that (H_2) is proved if

$$
\overline{\lim_{s\to 1}}\, \overline{\lim_{\rho\to 0}}\, \overline{\lim_{\varepsilon\to 0}} \frac{m_\varepsilon(w; \Sigma_\rho(x))}{\rho^d} = 0. \tag{5.14}
$$

Indeed, passing to the limits $\varepsilon \to 0$, $\rho \to 0$, $s \to 1$ in (5.13) we have

$$
\begin{aligned}
\overline{\lim_{\rho\to 0}}\, \underline{\lim_{\varepsilon\to 0}} \frac{F_\varepsilon(u+\varphi_\varepsilon; Q_\rho(x))}{\rho^d} &\geqslant \overline{\lim_{s\to 1}}\, \overline{\lim_{\rho\to 0}} \frac{m_-(u_x; Q_{s\rho}(x))}{(s\rho)^d} \\
&\geqslant \overline{\lim_{\rho\to 0}} \frac{m_-(u_x; Q_\rho(x))}{\rho^d}.
\end{aligned} \tag{5.15}
$$

To verify the last inequality in (5.15), we write by using Lemma 6.2

$$
\begin{aligned}
m_\varepsilon(u_x; Q_{s\rho}(x)) &= m_\varepsilon(u_x; Q_\rho(x)) + m_\varepsilon(u_x; \Sigma_\rho(x)) \\
&\geqslant m_\varepsilon(u_x; Q_\rho(x)),
\end{aligned}
$$

dividing by ρ^d and taking the $\underline{\lim}$ as $\varepsilon \to 0$ we have

$$
s^d \frac{m_-(u_x; Q_{s\rho}(x))}{(s\rho)^d} \geqslant \frac{m_-(u_x; Q_\rho(x))}{\rho^d}.
$$

Passing to the limit $\rho \to 0$ and then $s \to 1$ we obtain the last inequality in (5.15).

So, it remains to prove (5.14). Using (P_1) we have for some $C > 0$ dependent on p only

$$
\begin{aligned}
&\frac{m_\varepsilon(w; \Sigma_\rho(x))}{\rho^d} \\
&\leqslant \beta \frac{1}{\rho^d} \int_{\Sigma_\rho(x)} |\phi(u+\varphi_\varepsilon) + (1-\phi) u_x|^p \, dy \\
&+ C\beta \left(\left(s^d - 1\right) |\nabla u(x)|^p + \frac{\nu(\Sigma_\rho(x))}{\rho^d} \right) \\
&+ C\beta \left(\frac{1}{\rho^d} \int_{\Sigma_\rho(x)} |\nabla u + \nabla \varphi_\varepsilon|^p + \frac{1}{\rho^d} \int_{\Sigma_\rho(x)} |\nabla\phi \otimes (u + \varphi_\varepsilon - u_x)|^p \, dy \right) \\
&\leqslant C\beta \left(\left(s^d - 1\right) |\nabla u(x)|^p + \frac{\nu(\Sigma_\rho(x))}{\rho^d} + \frac{1}{\alpha} \frac{1}{\rho^d} F_\varepsilon(u+\varphi_\varepsilon; \Sigma_\rho(x)) \right) \\
&+ C\beta \frac{2^{3p-1}}{(s-1)^p} \left(s^{d+p} \frac{1}{(s\rho)^p} \fint_{Q_{s\rho}(x)} |u_x - u|^p \, dy - \frac{1}{\rho^p} \fint_{Q_\rho(x)} |u_x - u|^p \, dy \right) \\
&+ C\beta \frac{2^{3p-1}}{(s-1)^p} \left(s^{d+p} \frac{1}{(s\rho)^p} \fint_{Q_{s\rho}(x)} |\varphi_\varepsilon|^p \, dy - \frac{1}{\rho^p} \fint_{Q_\rho(x)} |\varphi_\varepsilon|^p \, dy \right) \\
&+ C\beta\rho^p \left(\frac{1}{\rho^p} \fint_{Q_\rho(x)} |u_x - u|^p dy - \frac{s^{d+p}}{(s\rho)^p} \fint_{Q_{s\rho}(x)} |u_x - u|^p dy \right) \\
&+ C\beta \left(\fint_{Q_\rho(x)} |u|^p - s^d \fint_{Q_{s\rho}(x)} |u|^p \right) + C\beta \frac{1}{\rho^d} \int_{\Sigma_\rho(x)} |\varphi_\varepsilon|^p \, dy.
\end{aligned} \tag{5.16}
$$

Using (5.10) and Alexandrov theorem we have

$$\begin{aligned}\overline{\lim_{\varepsilon\to 0}}\frac{1}{\rho^d}F_\varepsilon\left(u+\varphi_\varepsilon;\Sigma_\rho\left(x\right)\right) &= \overline{\lim_{\varepsilon\to 0}}\frac{\mu_\varepsilon\left(\Sigma_\rho\left(x\right)\right)}{\rho^d}\\ &\leqslant \overline{\lim_{\varepsilon\to 0}}\frac{\mu_\varepsilon\left(\overline{\Sigma}_\rho\left(x\right)\right)}{\rho^d}\\ &\leqslant \frac{\mu_0\left(\overline{\Sigma}_\rho\left(x\right)\right)}{\rho^d}\\ &\leqslant s^d\frac{\mu_0\left(\overline{\mathrm{Q}}_{s\rho}\left(x\right)\right)}{\left(s\rho\right)^d}-\frac{\mu_0\left(\mathrm{Q}_\rho\left(x\right)\right)}{\rho^d}.\end{aligned}$$

Letting $\rho\to 0$ we deduce by using (5.11)

$$\overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\frac{1}{\rho^d}F_\varepsilon\left(u+\varphi_\varepsilon;\Sigma_\rho\left(x\right)\right)\leqslant\left(s^d-1\right)D_\lambda\mu_0\left(x\right). \tag{5.17}$$

Taking (5.2), (5.3), (5.4), (5.5) and (5.17) into account and passing to the limits $\varepsilon\to 0$ then $\rho\to 0$ in (5.16) we obtain

$$\overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\frac{m_\varepsilon\left(w;\Sigma_\rho\left(x\right)\right)}{\rho^d}\leqslant C\beta\left(s^d-1\right)\left(D_\lambda\nu\left(x\right)+|u\left(x\right)|^p+|\nabla u\left(x\right)|^p+D_\lambda\mu_0\left(x\right)\right)$$

since $\varphi_\varepsilon\to 0$ in $L^p\left(\Omega;\mathbb{R}^m\right)$ as $\varepsilon\to 0$. Passing to the limit $s\to 1$ we finally proved (5.14). ■

As an illustration of Theorem 5.1 we give two elementary examples.

Example 5.1 (Integrands "almost" nondecreasing). *For each $\varepsilon>0$ we consider $L_\varepsilon:\Omega\times\mathbb{R}^d\times\mathbb{M}^{m\times d}\to[0,\infty[$ a Borel measurable function such that*

(P$_2$) $\exists\gamma\geqslant 0\quad\exists\delta>0\quad\forall\varepsilon>0\quad\forall\eta\in]0,\varepsilon[\quad\forall\left(x,v,\xi\right)\in\Omega\times\mathbb{R}^d\times\mathbb{M}^{m\times d}$

$$L_\varepsilon\left(x,v,\xi\right)\leqslant L_\eta\left(x,v,\xi\right)+\gamma|\varepsilon-\eta|^\delta.$$

Note that if $\gamma=0$ then $\varepsilon\mapsto L_\varepsilon\left(\cdot,\cdot,\cdot\right)$ is nondecreasing when ε is decreasing.

We define $F_\varepsilon:L^p\left(\Omega;\mathbb{R}^m\right)\times\mathcal{O}\left(\Omega\right)\to[0,\infty]$ by

$$F_\varepsilon\left(u;O\right):=\int_O L_\varepsilon\left(x,u\left(x\right),\nabla u\left(x\right)\right)dx.$$

Then it is direct to see that (H$_1$) *holds. If we assume* (5.1) *then* (P$_1$) *holds.*

Example 5.2 (Constant integrands with perturbation). *Let $W:\Omega\times\mathbb{R}^d\times\mathbb{M}^{m\times d}\to[0,\infty[$ be a Borel measurable integrand satisfying p-growth and p-coercivity* (5.1)*. Let $\{\Phi_\varepsilon\}_\varepsilon\subset L^1\left(\Omega;\mathbb{R}^+\right)$ such that*

(*i*) *there exists $g\in L^1\left(\Omega\right)$ such that $\Phi_\varepsilon\left(x\right)\leqslant g\left(x\right)$ for all $x\in\Omega$ and all $\varepsilon>0$;*

(*ii*) *there exists a nonnegative Borel measure Φ_0 such that*

$$\Phi_\varepsilon\left(\cdot\right)\lambda\overset{*}{\rightharpoonup}\Phi_0\qquad\text{as }\varepsilon\to 0. \tag{5.18}$$

For each $\varepsilon>0$ we set $L_\varepsilon:\Omega\times\mathbb{R}^d\times\mathbb{M}^{m\times d}\to[0,\infty[$ defined by

$$L_\varepsilon\left(x,v,\xi\right)=W\left(x,v,\xi\right)+\Phi_\varepsilon\left(x\right).$$

Then for each $O\in\mathcal{O}\left(\Omega\right)$ the family $\mathcal{F}\left(\cdot;O\right)$ $\Gamma(L^p)$-converges to

$$\mathcal{F}_0\left(u;O\right)=\int_O W_0\left(x,u\left(x\right),\nabla u\left(x\right)\right)+D_\lambda\Phi_0\left(x\right)dx.$$

Indeed, we have that (P$_1$) *holds because of the p-growth of W and* (*i*)*. Now, we have for almost all $x\in\Omega$*

$$\lim_{\rho\to 0}\lim_{\varepsilon\to 0}\fint_{\mathrm{Q}_\rho(x)}\Phi_\varepsilon\left(y\right)dy=D_\lambda\Phi_0\left(x\right)$$

since (5.18). *We can see that for every* $x \in \Omega$, *every* $\varepsilon > 0$, *every* $\rho > 0$ *and every* $u \in W^{1,p}(\Omega;\mathbb{R}^m)$

$$\inf\left\{ \fint_{Q_\rho(x)} W(y, w(y), \nabla w(y))\, dy : w \in u_x + W_0^{1,p}(Q_\rho(x);\mathbb{R}^m) \right\}$$
$$+ \fint_{Q_\rho(x)} \Phi_\varepsilon(y)\, dy.$$

It means that $(\mathscr{H})$ *holds and Theorem 5.2 applies with*

$$L_0(x, u(x), \nabla u(x)) = W_0(x, u(x), \nabla u(x)) + D_\lambda \Phi_0(x).$$

We give a concrete example. Assume that $\Omega = B_1(0) \subset \mathbb{R}^d$ *the euclidean open ball with center* 0 *and radius* 1. *Let* $g : \Omega \to [0,\infty]$ *be defined by*

$$g(x) := \begin{cases} \frac{2}{\sqrt{\|x\|}} & \text{if } x \in \Omega\backslash\{0\} \\ +\infty & \text{if } x = 0. \end{cases}$$

where $\|\cdot\|$ *is the euclidean norm. Then* $g \in L^1(\Omega)$. *For each* $\varepsilon > 0$ *we set for every* $x \in \Omega$

$$\Phi_\varepsilon(x) := \frac{1}{\sqrt{\varepsilon}} \mathbb{1}_{B_\varepsilon(0)}(x) + h(x)$$

where $h \in L^1(\Omega)$ *and satisfies* $h(x) \leqslant \frac{1}{2} g(x)$ *for all* $x \in \Omega$. *Then (i) and (ii) hold with*

$$\Phi_\varepsilon(\cdot)\lambda \overset{*}{\rightharpoonup} \Phi_0 := \delta_0 + h\lambda \quad \text{as } \varepsilon \to 0$$

where δ_0 *is the dirac measure at* 0. *It follows that*

$$D_\lambda \Phi_0(x) = h(x) \text{ a.e. in } \Omega.$$

5.2. **Homogenization.** Let $L : \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0,\infty[$ be a Borel measurable function which is p-coercive, i.e., there exists $\alpha > 0$ such that

$$\alpha|\xi|^p \leqslant L(x,\xi)$$

for all $(x,\xi) \in \Omega \times \mathbb{M}^{m\times d}$. For each $\varepsilon > 0$ we consider $F_\varepsilon : W^{1,p}(\Omega;\mathbb{R}^m) \times \mathcal{O}(\Omega) \to [0,\infty]$ given by

$$F_\varepsilon(u;O) := \int_O L\left(\frac{x}{\varepsilon}, \nabla u(x)\right) dx.$$

The family $\mathcal{F} = \{F_\varepsilon\}_\varepsilon \subset \mathcal{I}(p,\alpha)$. For each $\xi \in \mathbb{M}^{m\times d}$ we define $\mathcal{S}_\xi : \mathcal{O}(\Omega) \to [0,\infty]$ a set function by

$$\mathcal{S}_\xi^L(O) := \inf\left\{ \int_O L(y, \xi + \nabla\varphi(y))\, dy : \varphi \in W_0^{1,p}(O;\mathbb{R}^m) \right\}.$$

Definition 5.1. *We say that* L *is an* H-integrand *(*H *stands for "homogenizable") if*

$$(\mathscr{H}) \ \forall \xi \in \mathbb{M}^{m\times d} \qquad \varlimsup_{\rho\to 0} \varlimsup_{t\to\infty} \frac{\mathcal{S}_\xi^L(tQ_\rho(x))}{\lambda(tQ_\rho(x))} = \varlimsup_{\rho\to 0} \varliminf_{t\to\infty} \frac{\mathcal{S}_\xi^L(tQ_\rho(x))}{\lambda(tQ_\rho(x))} \quad \text{a.e. in } \Omega.$$

In this case we denote the common value by $L_{\rm hom}(x,\xi)$.

We see that $(\mathscr{H})$ implies (H_1), indeed, for every $u \in \mathrm{M}_{\mathcal{F}}(O)$ we have

$$\frac{\mathcal{S}^L_{\nabla u(x)}\left(\frac{1}{\varepsilon} Q_\rho(x)\right)}{\lambda\left(\frac{1}{\varepsilon} Q_\rho(x)\right)} = \frac{m_\varepsilon(u_x \cdot; Q_\rho(x))}{\rho^d}.$$

for all $\varepsilon > 0$ and all $x \in O$. So, we can deduce from Theorem 5.1 the following result.

Theorem 5.2. *If* (P_1) *holds and* L *is an* H*-integrand, i.e.,* $(\mathscr{H})$ *holds. Then for each* $O \in \mathcal{O}(\Omega)$ *the family* $\mathcal{F}(\cdot;O)$ $\Gamma(L^p)$*-converges at every* $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ *to*

$$\mathcal{F}_0(u,O) = \int_O L_{\rm hom}(x, \nabla u(x))\, dx$$

where

$$L_{\rm hom}(x,\xi) = L_0(x,\xi) = \lim_{\rho\to 0} \varlimsup_{t\to\infty} \frac{\mathcal{S}_\xi^L(tQ_\rho(x))}{\lambda(tQ_\rho(x))}.$$

for all $x \in O$ *and* $\xi \in \mathbb{M}^{m\times d}$.

Theorem 5.2 becomes a “classical” homogenization result when L_{hom} does not depend on x. For instance, when L is 1-periodic or almost periodic with respect to the first variable then by subadditive theorems [LM02, Theorem 2.1 and Theorem 3.1] the condition $(\mathscr{H})$ holds, i.e., L is an H-integrand, and we have

$$L_{\text{hom}}(\xi) = \inf_{n\in\mathbb{N}} \frac{\mathcal{S}_\xi^L(nY)}{n^d} \qquad \text{(periodic case)} \tag{5.19}$$

$$L_{\text{hom}}(\xi) = \lim_{n\to\infty} \frac{\mathcal{S}_\xi^L(nY)}{n^d} \qquad \text{(almost-periodic case).} \tag{5.20}$$

Example 5.3 (Periodic integrand with perturbation)**.** *Consider $W : \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0,\infty[$ be a Borel measurable function 1-periodic with respect to the first variable, i.e.,*

$$\forall x \in \mathbb{R}^d \quad \forall z \in \mathbb{Z}^d \quad \forall \xi \in \mathbb{M}^{m\times d} \qquad W(x+z,\xi) = W(x,\xi),$$

and satisfying p-growth and p-coercivity, i.e, there exist $\alpha,\beta>0$ such that

$$\forall (x,\xi) \in \mathbb{R}^d \times \mathbb{M}^{m\times d} \qquad \alpha|\xi|^p \leqslant W(x,\xi) \leqslant \beta(1+|\xi|^p).$$

Let $\Phi \in L^1_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^+)$ such that

(*i*) *there exists $g \in L^1_{\text{loc}}(\mathbb{R}^d)$ such that $\Phi\left(\frac{x}{\varepsilon}\right) \leqslant g(x)$ for all $x\in\Omega$ and all $\varepsilon>0$;*

(*ii*) *there exists a nonnegative Borel measure Φ_0 such that*

$$\Phi\left(\frac{\cdot}{\varepsilon}\right)\lambda \overset{*}{\rightharpoonup} \Phi_0 \quad \text{as } \varepsilon \to 0. \tag{5.21}$$

Let $L : \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0,\infty[$ be defined by

$$L(x,\xi) = W(x,\xi) + \Phi(x).$$

Note that L is not periodic with respect to the first variable, because of the “perturbation” Φ.

We consider the family $\mathcal{F} = \{F_\varepsilon\}_\varepsilon \subset \mathcal{I}(p,\alpha)$ given by

$$F_\varepsilon(u;O) := \int_O L\left(\frac{x}{\varepsilon}, \nabla u(x)\right) dx$$

for all $(u,O) \in W^{1,p}(\Omega;\mathbb{R}^m) \times \mathcal{O}(\Omega)$. Then $\mathcal{F}(\cdot;O)$ $\Gamma(L^p)$-converges to

$$\mathcal{F}_0(u;O) = \int_O W_{\text{hom}}(\nabla u(x)) + D_\lambda \Phi_0(x)\, dx$$

for all $u \in W^{1,p}(\Omega;\mathbb{R}^m) \times \mathcal{O}(\Omega)$, and where $W_{\text{hom}}(\xi)$ is given by the formula (5.19) *with $\mathcal{S}_\xi^W$ in place of $\mathcal{S}_\xi^L$. Indeed,* (P$_1$) *holds because of the p-growth of W and* (*i*)*. Now, we have for almost all $x\in\Omega$*

$$\lim_{\rho\to 0}\lim_{\varepsilon\to 0} \fint_{Q_\rho(x)} \Phi\left(\frac{y}{\varepsilon}\right) dy = D_\lambda\Phi_0(x)$$

since (5.21)*. Using* [LM02, Theorem 2.1] *we have for every $\xi \in \mathbb{M}^{m\times d}$*

$$W_{\text{hom}}(\xi) + D_\lambda\Phi_0(x) = \overline{\lim_{\rho\to 0}}\ \overline{\lim_{t\to\infty}} \frac{\mathcal{S}_\xi^L(tQ_\rho(x))}{\lambda(tQ_\rho(x))} = \overline{\lim_{\rho\to 0}}\ \underline{\lim_{t\to\infty}} \frac{\mathcal{S}_\xi^L(tQ_\rho(x))}{\lambda(tQ_\rho(x))} \quad \text{a.e. in } \Omega,$$

since we can see that for every $x\in\Omega$, every $t>0$ and every $\rho>0$

$$\frac{\mathcal{S}_\xi^L(tQ_\rho(x))}{\lambda(tQ_\rho(x))} = \inf\left\{\fint_{tQ_\rho(x)} W(y,\xi+\nabla\varphi(y))\,dy : \varphi \in W_0^{1,p}(tQ_\rho(x);\mathbb{R}^m)\right\} + \fint_{tQ_\rho(x)} \Phi(y)\,dy.$$

It means that L is an H*-integrand and Theorem 5.2 apply with*

$$L_{\text{hom}}(x,\xi) = W_{\text{hom}}(\xi) + D_\lambda\Phi_0(x).$$

Remark 5.1. An interesting problem in the field of deterministic homogenization (see [NNW10]) is the characterization of all H-integrands $L : \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0,\infty[$ Borel measurable with p-growth and p-coercivity, i.e., satisfying

$$\exists\alpha>0 \quad \exists\beta>0 \quad \forall\,(x,\xi)\in\Omega\times\mathbb{M}^{m\times d}$$

$$\alpha|\xi|^p \leqslant L(x,\xi) \leqslant \beta\left(1+|\xi|^p\right).$$

5.3. **Relaxation.** The following result is an extension of Acerbi-Fusco-Dacorogna relaxation theorem (see [Dac08, Theorem 9.8, p. 432] and [AF84, Statement III.7, p. 144]) in the case where the integrand is assumed Borel measurable only.

Theorem 5.3. *If $L : \Omega\times\mathbb{R}^m\times\mathbb{M}^{m\times d}\to[0,\infty[$ is Borel measurable and satisfies* (5.1) *then for every $O\in\mathcal{O}(\Omega)$*

$$\mathcal{F}_0(u;O) = \int_O L_0\left(x,u(x),\nabla u(x)\right)dx$$

where for a.a. $x\in O$

$$\begin{aligned}&L_0\left(x,u(x),\nabla u(x)\right)\\&=\lim_{\rho\to 0}\inf\left\{\fint_{\mathrm{Q}_\rho(x)} L\left(y,w(y),\nabla w(y)\right)dy : w\in u_x+W_0^{1,p}\left(\mathrm{Q}_\rho(x);\mathbb{R}^m\right)\right\}.\end{aligned}$$

Moreover, if L is Carathéodory, i.e.,

(i) for each $(v,\xi)\in\mathbb{R}^m\times\mathbb{M}^{m\times d}$ the function $\Omega\ni x\mapsto L(x,v,\xi)$ is measurable;
(ii) for a.a. $x\in\Omega$ the function $\mathbb{R}^m\times\mathbb{M}^{m\times d}\ni(v,\xi)\mapsto L(x,v,\xi)$ is continuous,

then for almost every $x\in\Omega$ and for every $(v,\xi)\in\mathbb{R}^d\times\mathbb{M}^{m\times d}$

$$\widetilde{L}_0(x,v,\xi) = \inf\left\{\int_Y L\left(x,v,\xi+\nabla\varphi(y)\right)dy : \varphi\in W_0^{1,\infty}\left(Y;\mathbb{R}^m\right)\right\}. \tag{5.22}$$

Proof. The formula (5.22) follows from Proposition 5.1. ∎

Remark 5.2. Under the same assumptions of Theorem 5.3 and using Proposition 2.2 we also have

$$\mathcal{F}_0(u;O) = \mathcal{F}_0^{\mathfrak{D}}(u;O) = \int_O L_0\left(x,u(x),\nabla u(x)\right)dx$$

for all $(u,O)\in W^{1,p}(\Omega;\mathbb{R}^m)\times\mathcal{O}(\Omega)$.

We can give an extension of $W^{1,p}$-quasiconvexity as follows.

Definition 5.2. *We say that a Borel measurable integrand $L : \Omega\times\mathbb{R}^d\times\mathbb{M}^{m\times d}\to[0,\infty]$ is $W^{1,p}$-quasiconvex if for every $(x,v,\xi)\in\Omega\times\mathbb{R}^d\times\mathbb{M}^{m\times d}$*

$$\widetilde{L}_0(x,v,\xi) = L(x,v,\xi).$$

However, when the integrand is dependent on (x,v) this generalization of quasiconvexity is more difficult to handle. When the integrand L is Carathéodory the variables x and v can be frozen and we recover the classical concept of quasiconvexity.

Proposition 5.1. *If L is Carathéodory and satisfies p-growth* (5.1) *then for a.a. $x\in\Omega$ and for every $(v,\xi)\in\mathbb{R}^m\times\mathbb{M}^{m\times d}$ we have*

$$\widetilde{L}_0(x,v,\xi) = \inf\left\{\int_Y L\left(x,v,\xi+\nabla\varphi(y)\right)dy : \varphi\in W_0^{1,\infty}\left(Y;\mathbb{R}^m\right)\right\}. \tag{5.23}$$

Proof. For each $(x, v, \xi) \in \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times d}$ we denote by $Q^{\mathrm{dac}}L(x, v, \xi)$ the right hand side of (5.23). For each $\rho \in]0, 1[$ we define $\Lambda_\rho, L_\rho : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times d} \to [0, \infty[$ by

$$\Lambda_\rho(x, v, \xi) := \inf\left\{ \int_Y L(x + \rho y, v + \rho(\xi y + \psi(y)), \xi + \nabla\psi(y))\, dy : \right.$$
$$\left. \psi \in W_0^{1,\infty}(Y; \mathbb{R}^m) \right\};$$
$$L_\rho(x, v, \xi) := \inf\left\{ \int_Y L(x + \rho y, v + \rho(\xi y + \varphi(y)), \xi + \nabla\varphi(y))\, dy : \right.$$
$$\left. \varphi \in W_0^{1,p}(Y; \mathbb{R}^m) \right\}.$$

It is easy to see, by a change of variables, that for a.a. $x \in \Omega$ and for every $(v, \xi) \in \mathbb{R}^m \times \mathbb{M}^{m\times d}$ we have

$$\overline{\lim_{\rho\to 0}}\, L_\rho(x, v, \xi) = \widetilde{L}_0(x, v, \xi). \tag{5.24}$$

It is enough to show that for a.a. $x \in \Omega$, for every $(v, \xi) \in \mathbb{R}^m \times \mathbb{M}^{m\times d}$ and every $\rho \in]0, 1[$ it hold

$$Q^{\mathrm{dac}}L(x, v, \xi) = \lim_{\rho\to 0} \Lambda_\rho(x, v, \xi) \tag{5.25}$$

$$\Lambda_\rho(x, v, \xi) = L_\rho(x, v, \xi). \tag{5.26}$$

Indeed, combining (5.24), (5.25) and (5.26) we obtain (5.23).

Proof of (5.25). Let $\delta > 0$. By Scorza-Dragoni theorem, there exists a compact set $K_\delta \subset \overline{Y}$ such that $\lambda(Y \backslash K_\delta) < \delta$ and $L\lfloor_{K_\delta \times (\mathbb{R}^m \times \mathbb{M}^{m\times d})}$ is continuous. Fix $(x, v, \xi) \in \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times d}$ such that

$$a(x) = \lim_{\rho\to 0} \fint_{Q_\rho(x)} a(y)\, dy = \lim_{\rho\to 0} \int_Y a(x + \rho y)\, dy < \infty. \tag{5.27}$$

We show first that $\overline{\lim}_{\rho\to 0} \Lambda_\rho(x, v, \xi) \leqslant Q^{\mathrm{dac}}L(x, v, \xi)$. Note that

$$Q^{\mathrm{dac}}L(x, v, \xi) \leqslant L(x, v, \xi) \leqslant \beta(a(x) + |v|^p + |\xi|^p) < \infty.$$

Let $\varepsilon > 0$. There exists $\psi \in W_0^{1,\infty}(Y; \mathbb{R}^m)$ such that

$$\int_Y L(x, v, \xi + \nabla\psi(y))\, dy \leqslant \varepsilon + Q^{\mathrm{dac}}L(x, v, \xi). \tag{5.28}$$

Fix $\rho \in]0, 1[$. Set $g_\rho(y) := L(x + \rho y, v + \rho(\xi y + \psi(y)), \xi + \nabla\psi(y))$ and $g_0(y) := L(x, v, \xi + \nabla\psi(y))$ for all $y \in Y$. Using (5.28) we have

$$\begin{aligned} \Lambda_\rho(x, v, \xi) &\leqslant \int_{K_\delta} g_\rho(y)\, dy + \int_{Y\backslash K_\delta} g_\rho(y)\, dy \\ &= \int_{K_\delta} g_\rho(y) - g_0(y)\, dy + \int_{Y\backslash K_\delta} g_\rho(y) - g_0(y)\, dy \\ &+ \int_Y g_0(y)\, dy \\ &\leqslant \int_{K_\delta} |g_\rho(y) - g_0(y)|\, dy + \int_{Y\backslash K_\delta} |g_\rho(y) - g_0(y)|\, dy \\ &+ \varepsilon + Q^{\mathrm{dac}}L(x, v, \xi). \end{aligned} \tag{5.29}$$

By using the p-growth (5.1) it easy to see that there exists C depending on β and p only such that

$$\max\{g_0(y), g_\rho(y)\} \leqslant C\left(a(x + \rho y) + |v|^p + |\xi|^p + |\psi(y)|^p + |\nabla\psi(y)|^p\right) \text{ a.e. in } Y. \tag{5.30}$$

By continuity of $L\lfloor_{K_\delta \times (\mathbb{R}^m \times \mathbb{M}^{m\times d})}$ we have $g_\rho(y) - g_0(y) \to 0$ a.e. in K_δ as $\rho \to 0$. Using the domination (5.30) we obtain by applying the Lebesgue dominated convergence theorem

$$\lim_{\rho\to 0} \int_{K_\delta} |g_\rho(y) - g_0(y)|\, dy = 0. \tag{5.31}$$

By (5.30) we have

$$\int_{Y\backslash K_\delta} |g_\rho(y) - g_0(y)|\, dy \tag{5.32}$$
$$\leqslant 2C\left(\int_{Y\backslash K_\delta} a(x+\rho y)\, dy + \delta\left(a(x) + |v|^p + |\xi|^p + \|\psi\|_\infty^p + \|\nabla\psi\|_\infty^p\right)\right).$$

Note that $\{Y \ni y \mapsto a(x+\rho y)\}_{\rho\in]0,1[}$ is uniformly integrable since (5.27). So, taking the supremum over ρ and passing to the limit $\delta \downarrow 0$ in (5.32) we find that

$$\lim_{\delta\downarrow 0} \sup_{\rho\in]0,1[} \int_{Y\backslash K_\delta} |g_\rho(y) - g_0(y)|\, dy = 0. \tag{5.33}$$

Taking (5.31) and (5.33) into account in (5.29) we find

$$\overline{\lim_{\rho\to 0}}\, \Lambda_\rho(x,v,\xi) \leqslant \varepsilon + Q^{\mathrm{dac}} L(x,v,\xi).$$

Now, we want to show that $\underline{\lim}_{\rho\to 0} \Lambda_\rho(x,v,\xi) \geqslant Q^{\mathrm{dac}} L(x,v,\xi)$. Consider a sequence $\{\rho_n\}_{n\in\mathbb{N}} \subset]0,1[$ such that

$$\underline{\lim_{\rho\to 0}}\, \Lambda_\rho(x,v,\xi) = \lim_{n\to\infty} \Lambda_{\rho_n}(x,v,\xi) \leqslant \beta\left(a(x) + |v|^p + |\xi|^p\right) < \infty$$

since p-growth conditions (5.1). Fix $n \in \mathbb{N}$. We can choose $\psi_n \in W_0^{1,\infty}(Y;\mathbb{R}^m)$ such that

$$\int_Y g_n(y)\, dy \leqslant \rho_n + \Lambda_{\rho_n}(x,v,\xi)$$

where $g_n(y) := L\left(x+\rho_n y, v + \rho_n\left(\xi y + \psi_{\rho_n}(y)\right), \xi + \nabla\psi_{\rho_n}(y)\right)$ for all $y \in Y$. Since p-coercivity, we can choose a subsequence (not relabelled) such that

$$\psi_n \to \psi_\infty \text{ in } L^p(Y;\mathbb{R}^m); \tag{5.34}$$

$$\nabla\psi_n \rightharpoonup \nabla\psi_\infty \text{ in } L^p\left(Y;\mathbb{M}^{m\times d}\right). \tag{5.35}$$

Fix $\delta > 0$ and choose a compact set $K_\delta \subset \overline{Y}$ such that $\lambda(Y\backslash K_\delta) < \delta$ and $L\lfloor_{K_\delta\times(\mathbb{R}^m\times\mathbb{M}^{m\times d})}$ is continuous. We have by Eisen convergence theorem [Eis79, p. 75] that

$$g_n(y) - L\left(x,v,\xi+\nabla\psi_n(y)\right) \to 0 \text{ in measure in } K_\delta. \tag{5.36}$$

We have

$$\int_Y g_n(y)\, dy \geqslant \int_{K_\delta} g_n(y) - L\left(x,v,\xi+\nabla\psi_n(y)\right) dy$$
$$+ \int_{Y\backslash K_\delta} g_n(y) - L\left(x,v,\xi+\nabla\psi_n(y)\right) dy + Q^{\mathrm{dac}} L(x,v,\xi).$$

Using growth conditions we have for a.a. $y \in Y$

$$|g_n(y) - L\left(x,v,\xi+\nabla\psi_n(y)\right)| \tag{5.37}$$
$$\leqslant 2C\left(a(x+\rho_n y) + a(x) + |v|^p + |\xi|^p + |\psi_n(y)|^p + |\nabla\psi_n(y)|^p\right).$$

By taking (5.36), (5.37), (5.34) and (5.35) into account we have

$$\lim_{n\to\infty} \int_{K_\delta} |g_n(y) - L\left(x,v,\xi+\nabla\psi_n(y)\right)|\, dy = 0$$

since Vitali convergence theorem. Using (5.37) and reasoning similarly as in the first part of the proof we have

$$\lim_{\delta\to 0} \sup_{n\in\mathbb{N}} \int_{Y\backslash K_\delta} |g_n(y) - L\left(x,v,\xi+\nabla\psi_n(y)\right)|\, dy = 0.$$

It follows that

$$\underline{\lim_{\rho\to 0}}\, \Lambda_\rho(x,v,\xi) = \underline{\lim_{n\to\infty}}\, \Lambda_{\rho_n}(x,v,\xi) \geqslant \underline{\lim_{n\to\infty}} \int_Y g_n(y)\, dy \geqslant Q^{\mathrm{dac}} L(x,v,\xi).$$

Proof of (5.26). Fix $(x,v,\xi) \in \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times d}$ and $\rho \in]0,1[$. We only need to prove that

$$L_\rho(x,v,\xi) \geqslant \Lambda_\rho(x,v,\xi). \tag{5.38}$$

Let $\varepsilon > 0$. There exists $\varphi_\varepsilon \in W_0^{1,p}(Y;\mathbb{R}^m)$ such that

$$L_\rho(x,v,\xi) + \varepsilon \geqslant \int_Y L(x+\rho y, v+\rho(\xi y+\varphi_\varepsilon(y)), \xi+\nabla\varphi_\varepsilon(y))\,dy.$$

There exists a sequence $\{\psi_n\}_{n\in\mathbb{N}} \subset W_0^{1,\infty}(Y;\mathbb{R}^m)$ such that $\psi_n \to \varphi_\varepsilon$ in $W^{1,p}(Y;\mathbb{R}^m)$, $\psi_n \to \varphi_\varepsilon$ a.e. in Y and $\nabla\psi_n \to \nabla\varphi_\varepsilon$ a.e. in Y as $n\to\infty$. Using growth conditions we have for some C depending on β and p only, for a.a. $y \in Y$ and for all $n\in\mathbb{N}$

$$\begin{aligned}&L(x+\rho y, v+\rho(\xi y+\psi_n(y)), \xi+\nabla\psi_n(y))\\ &\leqslant C\left(a(x+\rho y)+|v|^p+|\xi|^p+|\psi_n(y)|^p+|\nabla\psi_n(y)|^p\right).\end{aligned}$$

Since L is Carathéodory we have

$$\begin{aligned}&\lim_{n\to\infty} L(x+\rho y, v+\rho(\xi y+\psi_n(y)), \xi+\nabla\psi_n(y))\\ &\qquad = L(x+\rho y, v+\rho(\xi y+\varphi_\varepsilon(y)), \xi+\nabla\varphi_\varepsilon(y)) \qquad \text{a.e. in } Y.\end{aligned}$$

Applying Vitali convergence theorem we obtain

$$\begin{aligned}\Lambda_\rho(x,v,\xi) &\leqslant \lim_{n\to\infty}\int_Y L(x+\rho y, v+\rho(\xi y+\psi_n(y)), \xi+\nabla\psi_n(y))\,dy\\ &= \int_Y L(x+\rho y, v+\rho(\xi y+\varphi_\varepsilon(y)), \xi+\nabla\varphi_\varepsilon(y))\,dy\\ &\leqslant L_\rho(x,v,\xi)+\varepsilon.\end{aligned}$$

Letting $\varepsilon\to 0$ we finally obtain (5.38). ■

6. Appendix

6.1. **Usage of Vitali covering theorem.** Let $A \subset O \in \mathcal{O}(\Omega)$ be a set which is not necessarily measurable. For each $x\in A$ we consider a family of closed balls $\mathcal{K}_x$ containing x of O satisfying $\inf\{\mathrm{diam}(\mathrm{Q}) : \mathrm{Q}\in\mathcal{K}_x\} = 0$ and $A \subset \cup_{\mathrm{Q}\in\mathcal{K}}\mathrm{Q}$ with $\mathcal{K} := \cup_{x\in A}\mathcal{K}_x$. We say that $\mathcal{K}$ is a fine cover of A.

Then there exists a countable pairwise disjointed family of balls $\{\overline{\mathrm{Q}}_i\}_{i\geqslant 1} \subset \mathcal{K}$ such that

$$\lambda\left(A \backslash \mathop{\cup}_{i=1}^{\infty} \mathrm{Q}_i\right) = 0.$$

It follows that for any $\mu \in \mathfrak{A}_\lambda(O)$, i.e. $\mu \ll \lambda\lfloor_O$, we have $\mu(A\backslash\cup_{i\geqslant 1}\mathrm{Q}_i) = 0$. Moreover, if $\lambda(A) < \infty$ then for any $\delta > 0$ we can choose a finite subfamily $\{\overline{\mathrm{Q}}_i\}_{i=1}^N \subset \mathcal{K}$ satisfying

$$\mu\left(A \backslash \mathop{\cup}_{i=1}^{N} \mathrm{Q}_i\right) < \delta.$$

6.2. **Level sets of derivative of set functions.** Let $G : \mathcal{Q}_o(\Omega) \to]-\infty,\infty]$ be a set function. Let $O \in \mathcal{O}(\Omega)$. For each $h\in\mathbb{R}$ we consider the strict sublevel (resp. superlevel) of the lower (resp. upper) derivative of G

$$S_h := \{x \in O : \underline{D}_\lambda G(x) < h\} \quad \left(\text{resp. } S^h := \left\{x\in O : \overline{D}_\lambda G(x) > h\right\}\right)$$

The following lemma give consequences of sublevel (resp. superlevel) sets of derivative of set functions.

Lemma 6.1. *Let $h\in\mathbb{R}$ and $\eta>0$. Then*

(*i*) *there exists a* countable *pairwise disjointed family $\{\mathrm{Q}_i\}_{i\in I} \subset \mathcal{Q}_o(O)$ such that*

$$\lambda\left(S_h \backslash \mathop{\cup}_{i\in I}\mathrm{Q}_i\right) = 0, \quad \forall i\in I \quad G(\mathrm{Q}_i) < h\lambda(\mathrm{Q}_i) \text{ and } \mathrm{diam}(\mathrm{Q}_i) \in]0,\eta[\tag{6.1}$$

(resp. $\lambda\left(S^h \backslash \mathop{\cup}_{i\in I}\mathrm{Q}_i\right) = 0, \quad \forall i\in I \quad G(\mathrm{Q}_i) > h\lambda(\mathrm{Q}_i)$ *and* $\mathrm{diam}(\mathrm{Q}_i) \in]0,\eta[$*);*

(*ii*) *for every $\delta>0$ there exists a* finite *pairwise disjointed family $\{\mathrm{Q}_i\}_{i\in I} \subset \mathcal{Q}_o(O)$ such that*

$$\lambda\left(S_h \backslash \mathop{\cup}_{i\in I}\mathrm{Q}_i\right) < \delta, \quad \forall i\in I \quad G(\mathrm{Q}_i) < h\lambda(\mathrm{Q}_i) \text{ and } \mathrm{diam}(\mathrm{Q}_i) \in]0,\eta[$$

(resp. $\lambda\left(S^h \backslash \mathop{\cup}_{i\in I}\mathrm{Q}_i\right) < \delta, \quad \forall i\in I \quad G(\mathrm{Q}_i) > h\lambda(\mathrm{Q}_i)$ *and* $\mathrm{diam}(\mathrm{Q}_i) \in]0,\eta[$*).*

Proof. Let $h \in \mathbb{R}$ and $\eta > 0$. We only give the proof for S_h, since similar arguments apply for S^h. Note that (ii) is a direct consequence of (i), so, we only show (i).

If $x \in S_h$ then for some $\varepsilon > 0$

$$\forall \rho \in]0, \eta[\qquad \inf \left\{ \frac{G(\mathrm{Q})}{\lambda(\mathrm{Q})} : \mathrm{Q} \in \mathfrak{B}_{x,\rho}(O) \right\} < h - \varepsilon$$

where $\mathfrak{B}_{x,\rho}(O) := \{\mathrm{Q} : x \in \mathrm{Q} \in \mathcal{Q}_o(O) \text{ and } \operatorname{diam}(\mathrm{Q}) \leqslant \rho\}$. For each $\rho \in]0, \eta[$ there exists $\mathrm{Q}_{x,\rho} \in \mathfrak{B}_{x,\rho}(O)$ such that

$$\frac{G(\mathrm{Q}_{x,\rho})}{\lambda(\mathrm{Q}_{x,\rho})} - \varepsilon \leqslant \inf \left\{ \frac{G(\mathrm{Q})}{\lambda(\mathrm{Q})} : \mathrm{Q} \in \mathfrak{B}_{x,\rho}(O) \right\} < h - \varepsilon. \tag{6.2}$$

Consider the family $\mathcal{K}_\eta := \left\{\overline{\mathrm{Q}_{x,\rho}}\right\}_{x \in S_h, \rho \in]0,\eta[}$ of closed cubes such that (6.2) holds. The family $\mathcal{K}_\eta$ is a fine cover of S_h, i.e.,

$$S_h \subset \underset{\mathrm{Q} \in \mathcal{K}_\eta}{\cup} \mathrm{Q} \quad \text{and} \quad \forall x \in S_h \quad \inf\{\operatorname{diam}(\mathrm{Q}) : \mathrm{Q} \in \mathcal{K}_{\eta,x}\} = 0$$

where $\mathcal{K}_{\eta,x} := \left\{\overline{\mathrm{Q}_{x,\rho}}\right\}_{\rho \in]0,\eta[} \subset \mathcal{K}_\eta$. By Vitali covering theorem we conclude (6.1). ■

6.3. **Proof of Lemma 3.1.** Fix $c \in \mathbb{R}$. We have to prove that

$$M_c := \{x \in O : \underline{D}_\lambda G(x) \leqslant c\}$$

is measurable. Fix $\eta > 0$. Set $h := c + \eta$. By Lemma 6.1 (i) there exists a countable pairwise disjointed family $\{\mathrm{Q}_i\}_{i \in I} \subset \mathcal{Q}_o(O)$ such that

$$\lambda\left(S_h \backslash \underset{i \in I}{\cup} \mathrm{Q}_i\right) = 0, \quad \forall i \in I \quad G(\mathrm{Q}_i) < h\lambda(\mathrm{Q}_i) \text{ and } \operatorname{diam}(\mathrm{Q}_i) \in]0, \eta[.$$

Since $S_h \supset M_c$ we have

$$\lambda\left(M_c \backslash \underset{i \in I}{\cup} \mathrm{Q}_i\right) = 0.$$

If we show that the Borel set $\mathrm{Q}^\infty := \cup_{i \in I} \mathrm{Q}_i \subset M_c$ then M_c will be the reunion of a Borel set and a λ-negligible set and so measurable since λ is complete. Let $z \in \mathrm{Q}^\infty$. Then there exists $i_z \in I$ such that $z \in \mathrm{Q}_{i_z}$. It follows that

$$\inf\left\{\frac{G(\mathrm{Q})}{\lambda(\mathrm{Q})} : z \in \mathrm{Q} \in \mathcal{Q}_o(\Omega),\ \operatorname{diam}(\mathrm{Q}) \leqslant \eta\right\} \leqslant \frac{G(\mathrm{Q}_{i_z})}{\lambda(\mathrm{Q}_{i_z})} \leqslant c + \eta.$$

Passing to the limit $\eta \to 0$ we obtain $\underline{D}_\lambda G(z) \leqslant c$ which means that $z \in M_c$. The proof is complete. ■

6.4. **Properties of the set function $\{m_\varepsilon(u; \cdot)\}_\varepsilon$.**

Lemma 6.2. *Let $(u, O) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega)$. Then the family $\{m_\varepsilon(u; \cdot)\}_\varepsilon$, $m_\varepsilon(u; \cdot) : \mathcal{O}(O) \to [0, \infty]$ satisfies*

(i) *for every $\varepsilon > 0$ and every $(U, V) \in \mathcal{O}(O) \times \mathcal{O}(O)$*

$$U \cap V = \varnothing \implies m_\varepsilon(u; U \cup V) = m_\varepsilon(u; U) + m_\varepsilon(u; V);$$

(ii) *for every $\varepsilon > 0$, every $U, V \in \mathcal{O}(O)$ with $U \subset V$*

$$\lambda(V \backslash U) = 0 \implies m_\varepsilon(u; U) = m_\varepsilon(u; V);$$

(iii) *In particular, for every $U \in \mathcal{O}(O)$ and $V \in \mathcal{O}(O)$ satisfying $U \subset V$ we have for every $\varepsilon > 0$*

$$\lambda(\partial U) = 0 \implies m_\varepsilon(u; V) = m_\varepsilon(u; U) + m_\varepsilon\left(u; V \backslash \overline{U}\right).$$

Proof. We recall that for $A \in \mathcal{O}(\Omega)$ we have

$$W_0^{1,p}(A; \mathbb{R}^m) = \left\{u \in W^{1,p}(\Omega; \mathbb{R}^m) : u = 0 \text{ in } \Omega \backslash A\right\}.$$

If $U, V \in \mathcal{O}(O)$ satisfy $U \cap V = \varnothing$ then for every $\varphi_i \in L^p(\Omega; \mathbb{R}^m)$ with $i \in \{0, 1, 2\}$ we have

$$\varphi_0 \in W_0^{1,p}(U \cup V; \mathbb{R}^m) \Longrightarrow \varphi_0 \mathbb{1}_U \in W_0^{1,p}(U; \mathbb{R}^m) \text{ and } \varphi_0 \mathbb{1}_V \in W_0^{1,p}(V; \mathbb{R}^m);$$

$$\varphi_1 \in W_0^{1,p}(U; \mathbb{R}^m) \text{ and } \varphi_2 \in W_0^{1,p}(V; \mathbb{R}^m) \Longrightarrow \varphi_1 \mathbb{1}_U + \varphi_2 \mathbb{1}_V \in W_0^{1,p}(U \cup V; \mathbb{R}^m).$$

Let $\varepsilon > 0$. To verify (i) it suffices to write for every $\varphi_1 \in W_0^{1,p}(U;\mathbb{R}^m)$ and $\varphi_2 \in W_0^{1,p}(V;\mathbb{R}^m)$

$$\begin{aligned} F_\varepsilon(u+\varphi_1;U) + F_\varepsilon(u+\varphi_2;V) &= F_\varepsilon(u+\varphi_1 \mathbb{1}_U + \varphi_2 \mathbb{1}_V; U \cup V) \\ &\geqslant m_\varepsilon(u;U\cup V), \end{aligned}$$

taking the infimum over φ_1 and φ_2 we obtain

$$m_\varepsilon(u;U) + m_\varepsilon(u;V) \geqslant m_\varepsilon(u;U\cup V).$$

We also have for every $\varphi_0 \in W_0^{1,p}(U\cup V;\mathbb{R}^m)$

$$\begin{aligned} F_\varepsilon(u+\varphi_0;U\cup V) &= F_\varepsilon(u+\varphi_0\mathbb{1}_U;U) + F_\varepsilon(u+\varphi_0\mathbb{1}_V;V) \\ &\geqslant m_\varepsilon(u;U) + m_\varepsilon(u;V). \end{aligned}$$

Taking the infimum over φ_0 we obtain

$$m_\varepsilon(u;U\cup V) \geqslant m_\varepsilon(u;U) + m_\varepsilon(u;V).$$

Then (i) is satisfied.

Consider $U, V \in \mathcal{O}(O)$ satisfying $U \subset V$ and $\lambda(V\backslash U) = 0$. Since $U \subset V$ we have $W_0^{1,p}(U;\mathbb{R}^m) \subset W_0^{1,p}(V;\mathbb{R}^m)$, thus $m_\varepsilon(u;U) \geqslant m_\varepsilon(u;V)$. Assume that $m_\varepsilon(u;V) < \infty$. For every $\eta > 0$ there exists $\varphi \in W_0^{1,p}(V;\mathbb{R}^m)$ such that $\infty > m_\varepsilon(u;V) + \eta \geqslant F_\varepsilon(u+\varphi;V)$. By using (C_2) we have

$$\begin{aligned} m_\varepsilon(u;V) + \eta \geqslant F_\varepsilon(u+\varphi;V) &= F_\varepsilon(u+\varphi\mathbb{1}_U;U) + F_\varepsilon(u+\varphi;V\backslash U) \\ &\geqslant m_\varepsilon(u;U). \end{aligned}$$

Note that $\varphi\mathbb{1}_U = \varphi$ a.e. in V and so $\varphi\mathbb{1}_U \in W_0^{1,p}(U;\mathbb{R}^m)$. Therefore (ii) is satisfied.

To prove (iii) it is sufficient to use the properties (ii), (i) together with the fact that we can write $V\backslash\left(U \cup \left(V\backslash\overline{U}\right)\right) = \partial U$ for all $U, V \in \mathcal{O}(O)$ satisfying $U \subset V$. ■

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