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Macroscopic Behavior of a randomly fibered medium

by

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Abstract

By using variational convergence combined with ergodic theory of subadditive processes, we describe the macroscopic behavior of a randomly fibered medium. The cross sections of the fibers are randomly distributed from a stationary point process, their size is of order ε while the stiffness of the material in the matrix is of order ε^p. The variational limit functional energy obtained when ε goes to 0 is deterministic and non local.

AMS *subject classifications*: 49J45, 74K15, 74C05, 74R20.

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Contents

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1 Introduction

We are interested in the determination of the macroscopic behavior of a randomly fibered structure whose reference configuration is the open subset $\mathcal{O} := \widehat{\mathcal{O}} \times (0,h)$ of $\mathbb{R}^3$, with basis $\widehat{\mathcal{O}} := (0,l_1) \times (0,l_2) \subset \mathbb{R}^2$. More precisely for $\varepsilon = \frac{1}{n}$ we consider the union of fibers $T_\varepsilon(\omega) := \varepsilon D(\omega) \times \mathbb{R}$ where $D(\omega) := \bigcup_{i\in\mathbb{N}} D(\omega_i)$ and $D(\omega_i)$ are disks distributed at random in $\mathbb{R}^2$ following a stochastic point process $\omega = (\omega_i)_{i\in\mathbb{N}}$ of $\mathbb{R}^2$ associated with a suitable probability space $(\Omega, \mathcal{A}, \mathbf{P})$ (for short we sometimes write T_ε instead of $T_\varepsilon(\omega)$) (see Figure 1 and Figures 3 and 4 in Section 2).

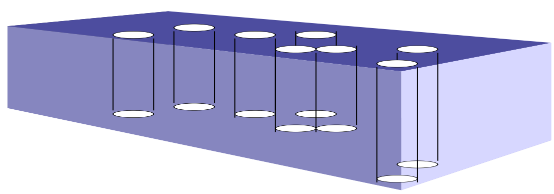

Figure 1: The random fibered structure

When ε goes to zero, we aim to obtain a deterministic equivalent variational limit of each three random integral functionals F_ε^v, G_ε and H_ε mapping $\Omega \times L^p(\mathcal{O}, \mathbb{R}^3)$ into $\mathbb{R}^+ \cup \{+\infty\}$, defined for every ω in $(\Omega, \mathcal{A}, \mathbf{P})$ by

$$F_\varepsilon^v(\omega, u) = \begin{cases} \varepsilon^p \displaystyle\int_{\mathcal{O}\setminus T_\varepsilon} f(\nabla u)\, dx \text{ if } u \in W^{1,p}_{\Gamma_0}(\mathcal{O}, \mathbb{R}^3),\ u = v \text{ on } \mathcal{O}\cap T_\varepsilon \\ +\infty \text{ otherwise,} \end{cases}$$

$$G_\varepsilon(\omega, u) = \begin{cases} \displaystyle\int_{\mathcal{O}\cap T_\varepsilon} g(\nabla_\varepsilon u)\, dx \text{ if } u \in W^{1,p}_{\Gamma_0}(T_\varepsilon \cap \mathcal{O}, \mathbb{R}^3),\ u = 0 \text{ in } \mathcal{O}\setminus T_\varepsilon \\ +\infty \text{ otherwise,} \end{cases}$$

$$H_\varepsilon(\omega, u) = \begin{cases} \varepsilon^p \displaystyle\int_{\mathcal{O}\setminus T_\varepsilon} f(\nabla u)\, dx + \int_{\mathcal{O}\cap T_\varepsilon} g(\nabla u)\, dx \text{ if } u \in W^{1,p}_{\Gamma_0}(\mathcal{O}, \mathbb{R}^3) \\ +\infty \text{ otherwise.} \end{cases}$$

The space $W^{1,p}_{\Gamma_0}(\mathcal{O}, \mathbb{R}^3)$ is the subset of all the functions u in $W^{1,p}(\mathcal{O}, \mathbb{R}^3)$ such that $u = 0$ on $\Gamma_0 := \widehat{\mathcal{O}} \times \{0\}$ in the trace sense, v is a given function in $W^{1,p}_{\Gamma_0}(\mathcal{O}, \mathbb{R}^3)$ and ∇_ε denotes the distributional gradient on $\mathcal{O}\cap T_\varepsilon$. For more precision on the stochastic point process $(\omega_i)_{i\in\mathbb{N}}$ and for all question of measurability relating to the considered random maps we refer the reader to the next section.

We assume that f and g are two quasiconvex functions defined on the set $\mathbf{M}^{3\times3}$ of 3×3-matrices and satisfy the standard growth condition of order $p > 1$: there exist two positive constants α, β, such that $\forall M, M' \in \mathbf{M}^{3\times3}$

$$\alpha|M|^p \leq f(M) \leq \beta(1 + |M|^p), \tag{1}$$

idem for g. Note that f satisfies automatically the Lipschitz property

$$|f(M) - f(M')| \leq L|M - M'|(1 + |M|^{p-1} + |M|^{p-1}) \tag{2}$$

for some positive constant L, idem for g. Furthermore, we assume that there exists $\beta' > 0$, $0 < \gamma < p$ and a p-positively homogeneous function $f^{\infty,p}$ (the p-recession function of f) such that for all $M \in \mathbf{M}^{3\times3}$

$$|f(M) - f^{\infty,p}(M)| \leq \beta'(1 + |M|^{p-\gamma}). \tag{3}$$

From (3) we infer $\lim_{t \to +\infty} \frac{f(tM)}{t^p} = f^{\infty,p}(M)$ so that from (1), $f^{\infty,p}$ satisfies for all $M \in \mathbf{M}^{3\times 3}$

$$\alpha |M|^p \le f^{\infty,p}(M) \le \beta |M|^p. \tag{4}$$

and

$$|f^{\infty,p}(M) - f^{\infty,p}(M')| \le L|M - M'|(|M|^{p-1} + |M|^{p-1}) \tag{5}$$

for all $(M, M') \in \mathbf{M}^{3\times 3} \times \mathbf{M}^{3\times 3}$.

As a consequence of the variational convergences we will provide an equivalent deterministic problem of

$$(\mathcal{P}_{H_\varepsilon}) \qquad \inf \left\{ H_\varepsilon(\omega, u) - \int_{\mathcal{O}} \mathcal{L}.u \; dx : u \in L^p(\mathcal{O}, \mathbb{R}^3) \right\}$$

where $\mathcal{L} \in L^q(\mathcal{O}, \mathbb{R}^3)$, $q = \frac{p}{p-1}$.

The functional G_ε models the internal energy of the mechanical structure made up of the union T_ε of thin parallel cylinders which represent the fibers clamped on $\widehat{\mathcal{O}}$. We only have a statistical knowledge of their cross sections of size ε in the sense that their positions are statistically homogeneous. From the mathematical point of view, this means that the cross sections are placed at random from a stationary point process. The functional F_ε^v models the internal energy of the mechanical body $\mathcal{O} \setminus T_\varepsilon$ clamped on $\widehat{\mathcal{O}}$, with a prescribed displacement along the random boundary $\partial T_\varepsilon \cap \mathcal{O}$; the stiffness of the soft elastic material occupying $\mathcal{O} \setminus T_\varepsilon$ is of order ε^p. We assume large deformations in the matrix and the fibers (see for instance [15]) so that the strong and soft materials are hyperelastic. Now assuming that the two bodies are perfectly stuck together and are subjected to an exterior loading $\mathcal{L}$, we derive the problem $(\mathcal{P}_{H_\varepsilon})$. Our objective is to analyze the behavior of $(\mathcal{P}_{H_\varepsilon})$ in a variational way when ε go to 0 and to provide a simplified but accurate model for the behavior of the slices of the geomaterial TexSol^{TM} ([12, 14, 15]). It is a soil reinforcement process created in 1984 by Leflaive, Khay and Blivet from the LCPC (Laboratoire Central des Ponts et Chaussées) which mixes the soil (sand) with a wire. The obtained reinforced material has a better mechanical resistance than the sand without wire. The wire is randomly distributed on the free surface and is covered with sand simultaneously to create a TexSol^{TM} layer. Although the wire volume is negligible compared to the sand one, the wire becomes a strong reinforcement when it tangles up inside sand. In our simplified model we assume the wire to cut the surface perpendicularly (the size h is small) so that the thin parallel cylinders, randomly distributed, represent the pieces of the wire which are perfectly stuck with a hyperelastic matrix which represent the sand (cf. Figure 2).

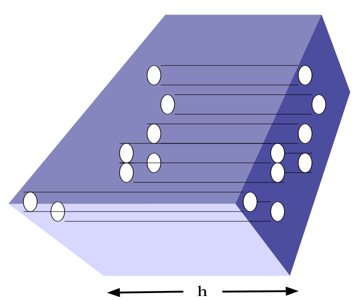

Figure 2: A slice of real material

From the mathematical point of view we reexamine the work of [5, 6, 17] in a stochastic setting. We establish the almost sure convergence of $(\mathcal{P}_{H_\varepsilon})$ when $\varepsilon \to 0$ to the deterministic and homogeneous problem

$$(\mathcal{P}_H) \qquad \min \left\{ H(u) - \int_{\mathcal{O}} \mathcal{L}.u \; dx : v \in L^p(\mathcal{O}, \mathbb{R}^3) \right\}$$

where the energy functional H is of non local nature. More precisely we establish the almost sure Γ-convergence of the sequence $(H_\varepsilon)_{\varepsilon>0}$ to the infimum convolution $F_0 \bigtriangledown G_0$ defined for every $u \in L^p(\mathcal{O},\mathbb{R}^3)$ by

$$F_0 \bigtriangledown G_0 \ (u) := \inf_{v\in L^p(\mathcal{O},\mathbb{R}^3)} \Big(F_0(u-v) + G_0(\mathbb{E}v)\Big)$$

(Theorem 5) where F_0 and G_0 are the functionals energy limits of the functionals energy associated with problems $(\mathcal{P}_{F_\varepsilon^v})$ and $(\mathcal{P}_{G_\varepsilon})$ respectively. They are defined in $L^p(\mathcal{O},\mathbb{R}^3)$ by

$$\begin{aligned} F_0(u) &= \int_{\mathcal{O}} f_0^{**}(u)\ dx,\\ f_0(a) &= \inf_{m\in\mathbb{N}^*} \left\{ \int_\Omega \frac{\mathcal{S}_{[0,m[^3}(\omega,a)}{m^3} d\mathbf{P}(\omega)\right\},\ a\in\mathbb{R}^3, \end{aligned}$$

where $\mathcal{S}$ is a suitable discrete subadditive process, and

$$G_0(u) = \begin{cases} \mathbb{E}\displaystyle\int_{\mathcal{O}} (g^\perp)^{**}(\frac{1}{\mathbb{E}}\ \frac{\partial u}{\partial x_3})dx \text{ if } \ u\in V_0\\ +\infty \text{ otherwise,}\end{cases}$$

where

$$\begin{aligned} & g^\perp(a) := \inf_{\xi\in\mathbf{M}^{3\times 2}} g\left(\xi|a\right),\ \mathbf{M}^{3\times 2} \text{ is the set of } 3\times 2 \text{ matrices,}\\ & V_0 := \Big\{ u\in L^p(\mathcal{O},\mathbb{R}^3) : \tfrac{\partial u}{\partial x_3} \in L^p(\mathcal{O},\mathbb{R}^3),\ u(\hat{x},0)=0 \text{ on } \widehat{\mathcal{O}} \Big\},\\ & \mathbb{E} = \int_\Omega |\hat{Y}\cap D(\omega)|\ d\mathbf{P}(\omega),\ \hat{Y}=(0,1)^2. \end{aligned}$$

In the deterministic case, i.e., when the fibers are periodically distributed, $\mathbb{E}$ reduces to $|\hat{Y}\cap D|$, and the density f_0^{**} to

$$f_0^{**}(a) = \inf\left\{ \int_{\hat{Y}\setminus D} (f^{\infty,p})^{**}(\hat{\nabla} w, 0)\ d\hat{y} : w\in W^{1,p}_\#(\hat{Y},\mathbb{R}^3),\ \int_{\hat{Y}} w\ d\hat{y} = a,\ w=0 \text{ in } D\right\}$$

where $W^{1,p}_\#(\hat{Y},\mathbb{R}^3)$ denotes the subset of $W^{1,p}(\hat{Y},\mathbb{R}^3)$ made up of $\hat{Y}$-periodic functions (Corollary 1).

Another interesting problem, perhaps more realistic, is to consider the case when the size of the fibers is very small compared with their density presence, with suitable adapted size of stiffness. The limit problem, seems to involve a non local model which takes into account the random capacity of $D(\omega)$ (see Remark 2). In this paper we do not address this situation which is examined in a work in progress.

2 The probabilistic framework

No difference is made between $\mathbb{R}^3$ and the three dimensional euclidean physical space equipped with an orthogonal basis denoted by (e_1,e_2,e_3). For all $x=(x_1,x_2,x_3)$ of $\mathbb{R}^3$, $\hat{x}$ stands for (x_1,x_2) and $\mathbf{M}^{3\times 3}$, $\mathbf{M}^{3\times 2}$ denotes the sets of 3×3 and 3×2 matrices. We denote by $\hat{Y}$ the unit cell $(0,1)^2$ of $\mathbb{R}^2$ and by Y the unit cell $(0,1)^3$ of $\mathbb{R}^3$.

For any $\delta>0$ and any non empty bounded set $\hat{A}$ of $\mathbb{R}^2$, we make use of the following notation: $\hat{A}_\delta := \Big\{x\in\hat{A} : d(x,\mathbb{R}^2\setminus\hat{A})>\delta\Big\}$. For any bounded Borel set A of $\mathbb{R}^2$ or $\mathbb{R}^3$, $|A|$ denotes its Lebesgue measure and $\#(A)$ its cardinal when it is finite.

Let $d>0$ be given small enough such that the open ball $\hat{B}_{d/2}(0)$ of $\mathbb{R}^2$ centered at 0 with radius $d/2$ is included in $(-\frac{1}{2},\frac{1}{2})^2$ and consider the set

$$\Omega = \left\{(\omega_i)_{i\in\mathbb{N}} : \omega_i\in\mathbb{R}^2,\ |\omega_i-\omega_j|\geq d \text{ for } i\neq j\right\},$$

equipped with the trace σ-algebra $\mathcal{A}$ of the standard product σ-algebra on Ω. For every $\omega=(\omega_i)_{(i)\in\mathbb{N}}$ we form the disk $D(\omega_i) := \omega_i + \hat{B}_{d/2}(0)$ and consider $D(\omega) := \bigcup_{i\in\mathbb{N}} D(\omega_i)$. Therefore $\omega\mapsto T(\omega) = D(\omega)\times\mathbb{R}$

is a random set in $\mathbb{R}^3$, union of random cylinders, whose basis is the union of the pairwise disjoint disks $D(\omega_i)$ of $\mathbb{R}^2$ centered at ω_i.

For every $z = (\hat{z}, z_3) \in \mathbf{Z}^3$ we define the operator $\tau_z : \Omega \to \Omega$ by $\tau_z \omega = \omega - \hat{z}$. Note that $T(\tau_z \omega) = T(\omega) - z$. Since τ_z does not depend on the component z_3 of z we sometimes write $\tau_{\hat{z}}$ instead of τ_z.

Furthermore we assume that there exists a probability measure on $(\Omega, \mathcal{A})$ which satisfies the system of three following axioms:

(A_1) *Non sparsely distribution*: $\mathbf{P}\Big(\Big\{ \omega \in \Omega : |\hat{Y} \cap D(\omega)| > 0 \Big\} \Big) = 1;$

(A_2) *Stationary condition*: $\forall z \in \mathbf{Z}^3$, $\tau_z \# \mathbf{P} = \mathbf{P}$ where $\tau_z \# \mathbf{P}$ denotes the probability image of $\mathbf{P}$ by τ_z;

(A_3) *Asymptotic mixing property*: for all sets E and F of $\mathcal{A}$, $\lim_{|z| \to +\infty} \mathbf{P}(\tau_z E \cap F) = \mathbf{P}(E)\mathbf{P}(F)$.

Remark 1. *i) It would be more natural to consider stationary condition (A_2) with respect to the continuous group $(\tau_t)_{t \in \mathbb{R}^3}$ defined in the same way by $\tau_t \omega = \omega - \hat{t}$. Actually the discrete group $(\tau_z)_{z \in \mathbf{Z}^3}$ suffices for the mathematical analysis. The size of the cell $\hat{Y}$ is chosen in such a way to fix the generator of the group $(\tau_z)_{z \in \mathbf{Z}^2}$. Condition (A_2) then says that every random function X taking its source in Ω is statistically homogeneous in the sense that X and $X \circ \tau_z$ have the same law (i.e. $X \# \mathbf{P} = X \circ \tau_z \# \mathbf{P}$). Roughly speaking, moving a window $\hat{A}$ in $\mathbb{R}^2$ following the translations in $\mathbb{R}^2$, the distributions of cross sections in the window are statistically the same.*

ii) Condition (A_1) together with condition (A_2) yield that the random set $D(\omega)$ is statistically not too sparse in $\mathbb{R}^2$. Indeed for every $\mathbf{Z}^2$-translated $\hat{A} = \hat{Y} + \hat{z}$ of $\hat{Y}$

$$\begin{aligned} \mathbf{P}\big(\Big\{ \omega : |\hat{A} \cap D(\omega)| > 0 \Big\} \big) &= \mathbf{P}\big(\Big\{ \omega : |\hat{Y} \cap (D(\omega) - \hat{z})| > 0 \Big\} \big) \\ &= \mathbf{P}\big(\Big\{ \omega : |\hat{Y} \cap (D(\tau_z \omega))| > 0 \Big\} \big) \\ &= \mathbf{P}\big(\Big\{ \omega : |\hat{Y} \cap (D(\omega))| > 0 \Big\} \big) = 1. \end{aligned}$$

iii) Condition (A_3) says that the events $\tau_z E$ and F are independent provided that z be large enough.

iv) Consider $\bar{\omega} = (\bar{\omega}_i)_{i \in \mathbb{N}}$ where $\bar{\omega}_i$ are the centers of the hexagonal close-packing of disks in $\mathbb{R}^2$. Then $\bar{\omega}$ is a "maximal" distribution in the sense that $|\hat{Y} \cap D(\omega)| \leq |\hat{Y} \cap D(\bar{\omega})|$ for a.s. ω in Ω.

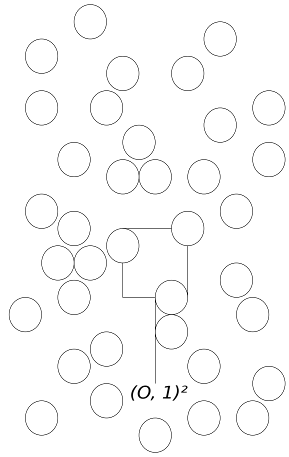

Figure 3: Random cross sections at scale $\varepsilon = 1$

A simple specimen of probability space which fulfills all the conditions above is the generalized random chessboard described bellow.

Example 1 (Random chessboard-like)**.** Given $d > 0$ as previously, let us consider a countable set of points $\Omega_0 = \{x_k : k \in \mathbb{N}\}$ in $\hat{Y}_{d/2}$ and set $\Omega := \Pi_{z \in \mathbf{Z}^2} \Omega_z$ where $\Omega_z = \Omega_0 + z$ for all $z \in \mathbf{Z}^2$. We equip Ω with the σ-algebra $\mathcal{A}$ generated by the cylinders of Ω. For a given family $(\alpha_k)_{k \in \mathbb{N}}$ of non negative numbers satisfying $\sum_{k \in \mathbb{N}} \alpha_k = 1$ we consider the probability measure $\mu_0 = \sum_{k \in \mathbb{N}} \alpha_k \delta_{x_k}$ on Ω_0 and the product probability measure $\mathbf{P} = \Pi_{z \in \mathbf{Z}} \mu_z$ on $(\Omega, \mathcal{A})$ where $\mu_z = \mu_0$ for all $z \in \mathbf{Z}$. Then it is easy to check that $\mathbf{P}$ satisfies axioms (A_1)-(A_3).

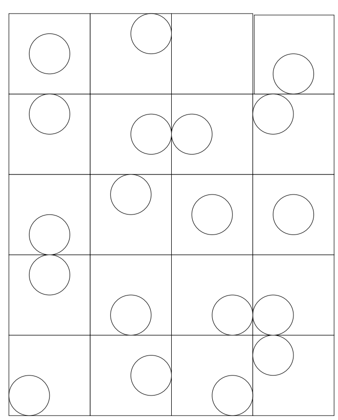

Figure 4: A piece of a random chessboard of cross sections at scale $\varepsilon = 1$ with $\#(\Omega_0) = 9$

Remark 2. *All the results of the paper remain valid if we substitute any connex compact set of* $\mathbb{R}^2$ *included in* $\hat{B}_{d/2}(0)$ *and chosen at random, for the disk* $\hat{B}_{d/2}(0)$*.*

We would stress that, keeping the same probabilistic framework, but substituting now $\hat{B}_{r_\varepsilon d/2}$, $r_\varepsilon \to 0$ *for* $\hat{B}_{d/2}$*, for critical growth of* $r(\varepsilon)$*, the limit problem seems to leads to a non local model which takes into account the random capacity of* $D(\omega)$*.*

Let us recall the following general basic notion of discrete subadditive process. We consider a probability space $(\Omega, \mathcal{A}, \mathbf{P})$ and a group $(\tau_s)_{s\in\mathbf{Z}^N}$ of $\mathbf{P}$-preserving transformations on $(\Omega, \mathcal{A})$. The group $(\tau_z)_{z\in\mathbf{Z}^N}$ is said to be ergodic if every set E in $\mathcal{A}$, such that $\tau_z E = E$ for every $z \in \mathbf{Z}^N$, satisfies $\mathbf{P}(E) = 0$ or $\mathbf{P}(E) = 1$. A sufficient condition to ensure ergodicity of $(\tau_z)_{z\in\mathbf{Z}^N}$ is the mixing condition (A_3): for every E and F in $\mathcal{A}$

$$\lim_{|z|\to+\infty} \mathbf{P}(\tau_z E \cap F) = \mathbf{P}(E)\mathbf{P}(F)$$

which expresses an asymptotic independence.

Let $\mathcal{I}$ denote the set of half open intervals $[a, b)$ of the lattice spanned by $(0, 1)^N$. A discrete subadditive process with respect to $(\tau_s)_{s\in\mathbf{Z}^N}$ is a set function $\mathcal{S} : \mathcal{I} \longrightarrow L^1(\Omega, \mathcal{A}, \mathbf{P})$ satisfying

(i) for every $I \in \mathcal{I}$ such that there exists a finite family $(I_j)_{j\in J}$ of disjoint intervals in $\mathcal{I}$ with $I = \bigcup_{j\in J} I_j$,

$$\mathcal{S}_I(\cdot) \le \sum_{j\in J} \mathcal{S}_{I_j}(\cdot),$$

(ii) $\forall I \in \mathcal{I},\ \forall z \in \mathbf{Z}^N$, $\mathcal{S}_I \circ \tau_z = \mathcal{S}_{z+I}$

A family $(I_n)_{n\in\mathbb{N}}$ of sets in $\mathcal{I}$ is called regular if there exists another family $(I'_n)_{n\in\mathbb{N}}$ of sets in $\mathcal{I}$ such that

(i) $I_n \subset I'_n$ for all $n \in \mathbb{N}$;

(ii) (I'_n) is non decreasing;

(iii) there exists a constant $C > 0$ such that $0 < |I'_n| \le C|I_n|$ for all $n \in \mathbb{N}$,

(iv) $\mathbb{R}^N_+ = \bigcup I'_n$.

Let $\mathbf{E}$ denote the expectation operator. The following subadditive ergodic theorem is due to Ackoglu-Krengel.

Theorem 1. *Let $\mathcal{S}$ be a discrete subadditive process with respect to an ergodic group $(\tau_s)_{s\in\mathbf{Z}^N}$ satisfying*

$$\inf\left\{\int_\Omega \frac{\mathcal{S}_I(\omega)}{|I|}\,\mathbf{P}(d\omega) I \in \mathcal{I},\ |I| \ne 0\right\} > -\infty$$

and let $(I_n)_{n\in\mathbb{N}}$ be a regular family of sets in $\mathcal{I}$. Then almost surely

$$\begin{aligned}\lim_{n\to\infty} \frac{\mathcal{S}_{I_n}}{|I_n|} &= \lim_{n\to\infty} \frac{\mathcal{S}_{[0,n[^N}}{n^N} \\ &= \inf_{n\in\mathbb{N}^*}\left\{\mathbf{E}\frac{\mathcal{S}_{[0,n[^N}}{n^N}\right\} = \lim_{n\to\infty} \mathbf{E}\frac{\mathcal{S}_{[0,n[^N}}{n^N}.\end{aligned}$$

For a proof see [1] and, for some extensions, see [13, 16].

We are going to define the limit density energy associated with the functional F^v_ε by applying Theorem 1 to a suitable subadditive process defined in the probabilistic space $(\Omega, \mathcal{A}, \mathbf{P})$ governed by axioms (A_1)-(A_3). For all $A \in \mathcal{I}$ and all $a \in \mathbb{R}^3$ set

$$\begin{aligned}\mathcal{S}_A(\omega, a) &:= \inf\left\{\int_{\overset{\circ}{A}\setminus\overline{T(\omega)}} f^{\infty,p}(\nabla w)\,dx : w \in \mathrm{Adm}_A(\omega, a)\right\}, \\ \mathrm{Adm}_A(\omega, a) &:= \left\{w \in W^{1,p}_0(\overset{\circ}{A}\setminus\overline{T(\omega)}, \mathbb{R}^3) : \fint_{\overset{\circ}{A}} w\,dx = a\right\}.\end{aligned}$$

Note that the random set $T(\omega)$ is not necessarily included in $\overset{\circ}{A}$. For every $w \in \mathrm{Adm}_A(\omega, a)$, we still denote by w its extension by zero in $T(\omega)\cap \overset{\circ}{A}$.

Since the Lebesgue measure does not charge the boundary of the elements of $\mathcal{I}$, one can take as $\mathcal{I}$ the set of all open intervals (a,b) of the lattice spanned by Y that we sometimes still denote by $\mathcal{I}$. Subsequently the subadditivity condition (i) becomes: for every $I \in \mathcal{I}$ such that there exists a finite family $(I_j)_{j\in J}$ of disjoint intervals in $\mathcal{I}$ with $|I \setminus \bigcup_{j\in J} I_j| = 0$,

$$\mathcal{S}_I(\cdot) \leq \sum_{j\in J} \mathcal{S}_{I_j}(\cdot).$$

It is standard to see that the random functionals defined in the introduction are measurable when $\Omega \times L^p(\mathcal{O},\mathbb{R}^3)$ is equipped with the product σ-algebra $\mathcal{A}\otimes\mathcal{B}$ where $\mathcal{B}$ is the Borel σ-algebra associated with the normed space $L^p(\mathcal{O},\mathbb{R}^3)$. Consequently, for all fixed A in $\mathcal{I}$ and all fixed a in $\mathbb{R}^3$, the map $\omega \mapsto \mathcal{S}_A(\omega,a)$ is measurable. Actually we have

Theorem 2. *For all fixed $a \in \mathbb{R}^3$, the map*

$$\begin{aligned} \mathcal{S}(.,a): \qquad \mathcal{I} &\longrightarrow L^1(\Omega,\mathcal{A},\mathbf{P}) \\ A &\longmapsto \mathcal{S}_A(.,a) \end{aligned}$$

is a subadditive process with respect to the group $(\tau_z)_{z\in\mathbf{Z}^3}$ defined by $\tau_z(\omega) = \omega - \hat{z}$. It satisfies for all $a \in \mathbb{R}^3$, all $A = \hat{A} \times A^\perp \in \mathcal{I}$ where $\hat{A} \subset \mathbb{R}^2$, $A^\perp \subset \mathbb{R}$, and all $\delta > 0$ small enough

$$\mathcal{S}_A(\omega,a) \leq C(p)\frac{C(A^\perp)}{\delta^p \left|(\hat{Y}\setminus D(\bar{\omega}))_{2\delta}\right|}|a|^p|A| \tag{6}$$

where $C(p)$ is a non negative constant depending only of p and $C(A^\perp)$ is a non negative constant depending only of $A^\perp$ and satisfying $C(A^\perp) \leq 1$ when the size of the interval $A^\perp$ is large enough.

Therefore for any regular family $(I_n)_{n\in\mathbb{N}}$ of sets in $\mathcal{I}$ the limit $\lim_{n\to\infty} \frac{\mathcal{S}_{I_n}(\omega,a)}{|I_n|}$ exists for $\mathbf{P}$ almost every $\omega \in \Omega$ and

$$\lim_{n\to\infty} \frac{\mathcal{S}_{I_n}(a,\omega)}{|I_n|} = \inf_{m\in\mathbb{N}^*}\left\{\mathbf{E}\frac{\mathcal{S}_{[0,m[^3}(.,a)}{m^3}\right\} = \lim_{m\to\infty}\left\{\mathbf{E}\frac{\mathcal{S}_{[0,m[^3}(.,a)}{m^3}\right\}.$$

We denote by f_0 the common value above.

Proof. We establish that $\mathrm{Adm}_A(\omega,a)$ is non empty and that $\mathcal{S}_A \in L^1(\Omega,\mathcal{A},\mathbf{P})$ by establishing (6). The rest of the proof consists in checking each condition (i) and (ii) and is straightforward. Let $A = \hat{A}\times A^\perp \in \mathcal{I}$. For $0 < \delta$ small enough consider $\phi_\delta = \rho_\delta * \mathbb{1}_{(\hat{A}\setminus D(\omega))_\delta}$ where ρ_δ is a standard mollifier. Clearly

$$\phi_\delta(\hat{x}) = \begin{cases} 1 \text{ if } \hat{x} \in (\hat{A}\setminus D(\omega))_{2\delta}, \\ 0 \text{ if } \hat{x} \in \mathbb{R}^2 \setminus (\hat{A}\setminus D(\omega)). \end{cases}$$

Therefore

$$-\!\!\!\!\!\!\int_{\hat{A}} \phi_\delta \, d\hat{x} \geq \frac{\left|(\hat{A}\setminus D(\omega))_{2\delta}\right|}{|\hat{A}|}.$$

Take $\bar{\omega}$ the close-packing distribution in $\mathbb{R}^2$ (Remark 1). According to (A_2), (A_3) we infer

$$\begin{aligned} -\!\!\!\!\!\!\int_{\hat{A}} \phi_\delta \, d\hat{x} \quad &\geq \quad \frac{\left|\sum_{z\in\hat{A}\cap\mathbf{Z}^2}(\hat{Y}+z\setminus D(\omega))_{2\delta}\right|}{|\hat{A}|} \\ &= \quad \frac{\left|\sum_{z\in\hat{A}\cap\mathbf{Z}^2}(\hat{Y}\setminus D(\tau_z\omega))_{2\delta}\right|}{|\hat{A}|} \\ &\geq \quad \frac{\#(\hat{A})}{|\hat{A}|}\left|(\hat{Y}\setminus D(\bar{\omega}))_{2\delta}\right| = \left|(\hat{Y}\setminus D(\bar{\omega}))_{2\delta}\right|. \end{aligned} \tag{7}$$

Take now $\theta \in \mathcal{C}_0^1(0,1)$ satisfying $\int_0^1 \theta(t)\ dt = 1$ and $\|\frac{d\theta}{dt}\|_\infty \leq C$. Without loss of generality we may assume $A^\perp = (0,m),\ m \in \mathbb{N}^*$. Set $\theta_{A^\perp} = \frac{1}{m}\theta(\frac{t}{m})$, then

$$\int_{A^\perp} \theta_{A^\perp}(s)\ ds = 1,\ \|\theta_{A^\perp}\|_\infty \leq \frac{C}{m},\ \|\frac{d\theta_{A^\perp}}{dt}\|_\infty \leq \frac{C}{m^2}.$$

The random function defined by $w_\delta(\hat{x}, x_3) = a\dfrac{\phi_\delta(\hat{x})\theta(x_3)}{\fint_A \phi_\delta\ d\hat{x}}$ clearly belongs to $adm_A(\omega, a)$ (for short we do note indicate the dependance on ω). Moreover from (7) and the growth condition satisfied by $f^{\infty,p}$,

$$\begin{aligned}
\mathcal{S}_A(\omega, a) &\leq \int_{A\setminus T(\omega)} f^{\infty,p}(\nabla w_\delta)\ d\hat{x} \\
&\leq C(p)\frac{C(A^\perp)}{\delta^p \left|((\hat{Y}\setminus D(\bar{\omega}))_{2\delta}\right|}|a|^p|A|.
\end{aligned}$$

where $C(p)$ is a non negative constant which depends only on p and $C(A^\perp) = \|\theta_{A^\perp}\|_\infty^p$. □

The elastic density associated with the limit internal energy of the material occupying $\mathcal{O}\setminus T_\varepsilon(\omega)$ is now defined for all $a \in \mathbb{R}^3$ by

$$\begin{aligned}
f_0^{**}(a) &= \left[\inf_{m\in\mathbb{N}^*}\left\{\mathbf{E}\frac{\mathcal{S}_{[0,m[^3}(.,.)}{m^3}\right\}\right]^{**}(a) \\
&= \left[\lim_{m\to\infty}\left\{\mathbf{E}\frac{\mathcal{S}_{[0,m[^3}(.,.)}{m^3}\right\}\right]^{**}(a)
\end{aligned}$$

where, for any function $h : \mathbb{R}^3 \to \mathbb{R}$, h^{**} stands for its convexification, i.e., the greater convex function less than h. In the deterministic case we have

Corollary 1. *Assume that the fibers are periodically distributed, i.e., in the chessboard-like example above, Ω_0 and Ω_1 are reduce to a single point, then for all $a \in \mathbb{R}^3$,*

$$f_0(a) = \inf_{n\in\mathbb{N}^*}\frac{\mathcal{S}_{(0,n)^3}(a)}{n^3}$$

where

$$\begin{aligned}
\mathcal{S}_A(a) &:= \inf\left\{\int_{\overset{\circ}{A}\setminus\overline{T}} f^{\infty,p}(\nabla w)\ d\hat{x} : w \in Adm_A(a)\right\}, \\
Adm_A(\omega, a) &:= \left\{w \in W_0^{1,p}(\overset{\circ}{A}\setminus\overline{T}, \mathbb{R}^3) : \fint_{\overset{\circ}{A}} w\ d\hat{x} = a\right\};
\end{aligned}$$

Furthermore $f_0^{**}(a)$ reduces to

$$f_0^{**}(a) = \inf\left\{\int_{\hat{Y}\setminus D}(f^{\infty,p})^{**}(\hat{\nabla}w, 0)\ d\hat{y} : w \in W_\#^{1,p}(\hat{Y}, \mathbb{R}^3),\ \int_{\hat{Y}} w\ d\hat{y} = a,\ w = 0 \ in\ D\right\}$$

where $W_\#^{1,p}(\hat{Y}, \mathbb{R}^3)$ denotes the subset of $W^{1,p}(\hat{Y}, \mathbb{R}^3)$ made up of $\hat{Y}$-periodic functions.

Proof. Clearly $f_0(a) = \inf_{n\in\mathbb{N}^*}\dfrac{\mathcal{S}_{(0,n)^3}(a)}{n^3}$. Thus for all $n \in \mathbb{N}^*$

$$f_0^{**}(a) \leq \Big(\frac{\mathcal{S}_{(0,n)^3}(.)}{n^3}\Big)^{**}(a)$$

so that

$$f_0^{**}(a) \leq \inf_{n\in\mathbb{N}^*}\Big(\frac{\mathcal{S}_{(0,n)^3}(.)}{n^3}\Big)^{**}(a).$$

Since the converse inequality is obviously satisfied, we conclude to

$$f_0^{**}(a) = \inf_{n\in\mathbb{N}^*} \Big(\frac{\mathcal{S}_{(0,n)^3}(.)}{n^3}\Big)^{**}(a). \tag{8}$$

But from standard arguments using Fenchel's Duality,

$$\Big(\frac{\mathcal{S}_{(0,n)^3}(.)}{n^3}\Big)^{**}(a) = \frac{1}{n^3} \inf \left\{ \int_{nY} (f^{\infty,p})^{**}(\nabla w)\ dy : w \in \mathrm{Adm}_{nY}(a) \right\}. \tag{9}$$

Let $w_\#$ be a minimizer of

$$\inf \left\{ \int_{Y\setminus T} (f^{\infty,p})^{**}(\nabla w)\ dy : w \in W^{1,p}_\#(Y,\mathbb{R}^3),\ \int_Y w\ dy = a,\ w = 0 \text{ in } T \right\},$$

extended by Y-periodicity on $\mathbb{R}^3$ and fix $n \in \mathbb{N}^*$. Clearly $\partial (f^{\infty,p})^{**}(\nabla w_\#(x))$ is non empty and for short, we assume that it is single valued. Note that $-\mathrm{div}\partial (f^{\infty,p})^{**}(\nabla w_\#) = 0$ a.e. in nY and $\partial (f^{\infty,p})^{**}(\nabla w_\#).\nu$ is anti-periodic, where ν denotes the unit normal to the boundary of nY. Take any $w \in \mathrm{Adm}_{nY}(a)$. According to the subdifferential inequality we have

$$\int_{nY} (f^{\infty,p})^{**}(\nabla w)\ dy \geq \int_{nY} (f^{\infty,p})^{**}(\nabla w_\#)\ dy + \int_{nY} \partial (f^{\infty,p})^{**}(\nabla w_\#).\nabla (w - w_\#)\ dy.$$

Integrating by parts, we infer

$$\int_{nY} (f^{\infty,p})^{**}(\nabla w)\ dy \geq \int_{nY} (f^{\infty,p})^{**}(\nabla w_\#)\ dy = n^3 \int_Y (f^{\infty,p})^{**}(\nabla w_\#)\ dy$$

so that from (9)

$$\Big(\frac{\mathcal{S}_{(0,n)^3}(.)}{n^3}\Big)^{**}(a) \geq \inf \left\{ \int_{Y\setminus T} (f^{\infty,p})^{**}(\nabla w)\ dy : w \in W^{1,p}_\#(Y,\mathbb{R}^3),\ \int_Y w\ dy = a,\ w = 0 \text{ in } T \right\}.$$

Thus, from (8) and since the converse inequality clearly holds

$$f_0^{**}(a) = \Big(\frac{\mathcal{S}_{(0,n)^3}(.)}{n^3}\Big)^{**}(a) = \inf \left\{ \int_{Y\setminus T} (f^{\infty,p})^{**}(\nabla w)\ dy : w \in W^{1,p}_\#(Y,\mathbb{R}^3),\ \int_Y w\ dy = a,\ w = 0 \text{ in } T \right\}.$$

The conclusion then follows by noticing that

$$\inf \left\{ \int_{Y\setminus T} (f^{\infty,p})^{**}(\nabla w)\ dy : w \in W^{1,p}_\#(Y,\mathbb{R}^3),\ \int_Y w\ dy = a,\ w = 0 \text{ in } T \right\}$$

is equal to

$$\inf \left\{ \int_{\hat{Y}\setminus D} (f^{\infty,p})^{**}(\hat{\nabla} w, 0)\ d\hat{y} : w \in W^{1,p}_\#(\hat{Y},\mathbb{R}^3),\ \int_{\hat{Y}} w\ d\hat{y} = a,\ w = 0 \text{ in } D \right\}$$

which is a straightforward consequence of Jensen's inequality. □

The following proposition is a straightforward consequence of estimate (6).

Proposition 1. *The function f_0^{**} is a positively homogeneous convex function of degree p and satisfies the growth conditions (4) and (5) where $\mathbf{M}^{3\times 3}$ is replaced by $\mathbb{R}^3$.*

We end this section by the following proposition which is a consequence of Theorem 1 when $\mathcal{S}$ is additive. It extends the Birkoff ergodic theorem.

Proposition 2. *Let $n \in \mathbb{N}^*$ and $\psi : \Omega \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}^2)$-measurable function satisfying the conditions:*

i) for $\mathbf{P}$-almost every $\omega \in \Omega$, $\hat{y} \mapsto \psi(\omega, \hat{y})$ belongs to $L^1_{loc}(\mathbb{R}^2)$;

ii) for all bounded Borel set $\hat{A}$ of $\mathbb{R}^2$ the map $\hat{A} \mapsto \int_{\hat{A}} \psi(\omega, \hat{y})\, d\hat{y}$ belongs to $L^1(\Omega, \mathcal{A}, \mathbf{P})$;

iii) for all $z \in n\mathbf{Z}^2$, for all $\hat{y} \in \mathbb{R}^2$, $\psi(\omega, \hat{y} + z) = \psi(\tau_z \omega, \hat{y})$ for $\mathbf{P}$-almost every $\omega \in \Omega$.

Then

$$x \mapsto \psi(\omega, \frac{\hat{x}}{\varepsilon}) \rightharpoonup x \mapsto \mathbf{E} \fint_{(0,n)^2} \psi(., \hat{y})\, d\hat{y}$$

for the $\sigma(L^1(\mathcal{O}), L^\infty(\mathcal{O}))$ topology.

Proof. See Theorem 4.2 and Proposition 5.3 in [8]. □

3 The limit problem associated with the soft material structure

Before studying the almost sure variational convergence of the functional F_ε we start by establishing a compactness result which explains why we equip $L^p(\mathcal{O}, \mathbb{R}^3)$ with its weak convergence. All along the paper we denote by $\to$ and $\rightharpoonup$ the strong and weak convergences in the various topological spaces, we do not relabel the subsequences and C will denote various nonnegative constants independent of ε and ω which may vary from line to line.

Lemma 1 (compactness)**.** *Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence satisfying* $\sup_{\varepsilon>0} F^v_\varepsilon(\omega, u_\varepsilon) < +\infty$ *for* $\mathbf{P}$ *a.s.* $\omega \in \Omega$. *Then for $\mathbf{P}$ a.s. $\omega \in \Omega$, there exists a subsequence possibly depending on ω and $u \in L^p(\mathcal{O}, \mathbb{R}^3)$ possibly depending on ω such that $u_\varepsilon \rightharpoonup u$ in $L^p(\mathcal{O}, \mathbb{R}^3)$.*

Proof. Fix $\omega \in \Omega$ such that (A_1) holds and such that $\sup_{\varepsilon>0} F^v_\varepsilon(\omega, u_\varepsilon) < +\infty$. Consider $w \in W^{1,p}(\mathbb{R}^2, \mathbb{R}^3)$. According to the Poincaré-Wirtinger inequality, there exists a constant $C(\omega)$ such that

$$\int_{\hat{Y}} \left| w - \fint_{\hat{Y} \cap D(\omega)} w\, d\hat{x} \right|^p d\hat{x} \leq C(\omega) \int_{\hat{Y}} |\nabla w|^p\, d\hat{x}$$

from which we easily deduce

$$\int_{\varepsilon\hat{Y}} \left| w - \fint_{\varepsilon\hat{Y} \cap \varepsilon D(\omega)} w\, d\hat{x} \right|^p d\hat{x} \leq C(\omega) \int_{\varepsilon\hat{Y}} |\varepsilon \nabla w|^p\, d\hat{x}$$

and finally

$$\int_{\varepsilon\hat{Y}} |w|^p \leq 2^p \Big(\varepsilon^2 \fint_{\varepsilon\hat{Y} \cap \varepsilon D(\omega)} |w|^p\, d\hat{x} + C(\omega) \int_{\varepsilon\hat{Y}} |\varepsilon \nabla w|^p\, d\hat{x} \Big). \tag{10}$$

Applying (10) to the function $\tau_{\varepsilon z} w$ defined by $\tau_{\varepsilon z} w(\hat{x}) := w(\hat{x} + \varepsilon z)$ we infer

$$\begin{aligned} \int_{\varepsilon(\hat{Y}+z)} |w|^p &= \int_{\varepsilon\hat{Y}} |\tau_{\varepsilon z} w|^p\, d\hat{x} \\ &\leq 2^p \Big(\varepsilon^2 \fint_{\varepsilon\hat{Y} \cap \varepsilon D(\omega)} |\tau_{\varepsilon z} w|^p\, d\hat{x} + C(\omega) \int_{\varepsilon\hat{Y}} |\varepsilon \nabla \tau_{\varepsilon z} w|^p\, d\hat{x} \Big) \\ &= 2^p \Big(\varepsilon^2 \fint_{\varepsilon(\hat{Y}+z) \cap \varepsilon D(\tau_{-z}\omega)} |w|^p\, d\hat{x} + C(\omega) \int_{\varepsilon(\hat{Y}+z)} |\varepsilon \nabla w|^p\, d\hat{x} \Big). \end{aligned} \tag{11}$$

Noticing that $\left|\widehat{\mathcal{O}} \setminus \bigcup_{z\in I_\varepsilon} \varepsilon(\hat{Y}+z)\right| = 0$ where I_ε is a finite subset of $\mathbf{Z}^2$ and $(\hat{Y}+z)_{z\in\mathbf{Z}^2}$ are pairwise disjoint, from (11), and since $u_\varepsilon = v$ on $\varepsilon D(\tau_{-z}\omega)\times(0,h)$, we obtain

$$\begin{aligned}
\int_{\mathcal{O}} |u_\varepsilon|^p\,dx &\leq 2^p\Big(\varepsilon^2 \sum_{z\in I_\varepsilon} \int_0^h \fint_{\varepsilon(\hat{Y}+z)\cap\varepsilon D(\tau_{-z}\omega)} |v|^p\,dx + C(\omega)\int_{\mathcal{O}} |\varepsilon\nabla u_\varepsilon|^p\,dx\Big)\\
&= C\Big(\sum_{z\in I_\varepsilon} \frac{1}{|(\hat{Y}+z)\cap D(\tau_{-z}\omega)|}\int_0^h \int_{\varepsilon(\hat{Y}+z)\cap\varepsilon D(\tau_{-z}\omega)} |v|^p\,dx + C(\omega)\int_{\mathcal{O}} |\varepsilon\nabla u_\varepsilon|^p\,dx\Big)\\
&\leq C\Big(\frac{1}{|\hat{Y}\cap D(\omega)|}\|v\|^p_{L^p(\mathcal{O},\mathbb{R}^3)} + C(\omega)\int_{\mathcal{O}} |\varepsilon\nabla u_\varepsilon|^p\,dx\Big)
\end{aligned}$$

and the conclusion follows from $\varepsilon^p \int_{\mathcal{O}\setminus T_\varepsilon} |\nabla u_\varepsilon|^p\,dx \leq \frac{1}{\alpha} F^v_\varepsilon(\omega,u_\varepsilon)$. □

Let us define the functional $F^v_0 : L^p(\mathcal{O},\mathbb{R}^3) \to \mathbb{R}^+$ by

$$F^v_0(u) = \int_{\mathcal{O}} f_0^{**}(u-v)\,dx$$

where f_0 is the function defined in Section 2. The following theorem is a consequence of the two bounds established in the next two subsections.

Theorem 3. *The sequence of random functionals $(F^v_\varepsilon)_{\varepsilon>0}$ almost surely sequentially Γ-converges to the deterministic functional F^v_0 when $L^p(\mathcal{O},\mathbb{R}^3)$ is equipped with its weak topology.*

3.1 The upper bound

Proposition 3. *There exists a set $\Omega' \in \mathcal{A}$ of full probability such that for all $u \in L^p(\mathcal{O},\mathbb{R}^3)$ and all $\omega \in \Omega'$ there exists a sequence $(u_\varepsilon(\omega))_{\varepsilon>0}$ in $L^p(\mathcal{O},\mathbb{R}^3)$ satisfying*

$$\begin{cases} u_\varepsilon(\omega) \rightharpoonup u \text{ in } L^p(\mathcal{O},\mathbb{R}^3)\\ F^v_0(u) \geq \limsup_{\varepsilon\to0} F^v_\varepsilon(\omega,u_\varepsilon(\omega)) \end{cases}$$

or, equivalently, for all $\omega \in \Omega'$, $\Gamma-\limsup F^v_\varepsilon(\omega,.) \leq F^v_0$.

Proof. We proceed into two steps.

Step 1. For every $v \in W^{1,p}(\mathcal{O},\mathbb{R}^3)$, let us consider the function $\tilde{F}^v_0 : L^p(\mathcal{O},\mathbb{R}^3) \to \mathbb{R}\cup\{+\infty\}$ defined by

$$\tilde{F}^v_0(u) = \begin{cases} \displaystyle\int_{\mathcal{O}} f_0(u-v)\,dx \text{ if } u \in \mathcal{C}^1_c(\mathcal{O},\mathbb{R}^3)\\ +\infty \text{ otherwise.}\end{cases}$$

We establish $\Gamma-\limsup F_\varepsilon(\omega,.) \leq \tilde{F}^v_0$ for $\mathbf{P}$ a.s. $\omega \in \Omega''$.

Let $\eta \in \mathbf{Q}^+$ intended to go to 0 and let $(Q_{i,\eta})_{i\in I_\eta}$ be a finite family of pairwise disjoint cubes of size η included in $\mathcal{O}$, such that

$$\left|\mathcal{O} \setminus \bigcup_{i\in I_\eta} Q_{i,\eta}\right| = 0.$$

Let $v_\delta \in \mathcal{C}^1_c(\mathcal{O},\mathbb{R}^3)$ be a regular approximation of v in $L^p(\mathcal{O},\mathbb{R}^3)$, i.e., satisfying $v_\delta \to v$ strongly in $L^p(\mathcal{O},\mathbb{R}^3)$, set $z_\delta := u - v_\delta$ and $z_{\delta,\eta} := \sum_{i\in I_\eta} z_\delta(x_{i,\eta})\mathbb{1}_{Q_{i,\eta}}$ where $x_{i,\eta}$ is arbitrarily chosen in $Q_{i,\eta}$. Since z_δ is a Lipschitz function on $\mathcal{O}$, clearly $z_{\delta,\eta} \to z_\delta = u - v_\delta$ in $L^p(\mathcal{O},\mathbb{R}^3)$ when $\eta \to 0$.

Consider the greater open cube $C_{i,\eta,\varepsilon}$ in $\mathcal{I}$ included in $\frac{1}{\varepsilon}\hat{Q}_{i,\eta}$ and let $w_{i,\eta,\varepsilon} \in adm_{C_{i,\eta,\varepsilon}}(\omega, z_\delta(x_{i,\eta}))$ be a ε-minimizer of $\mathcal{S}_{C_{i,\eta,\varepsilon}}(\omega, z_\delta(x_{i,\eta}))$ extended by zero outside $C_{i,\eta,\varepsilon} \setminus T(\omega)$ (for shorten notation, we do not indicate the dependance on δ). The family $(C_{i,\eta,\varepsilon})_\varepsilon$ is regular. Indeed for every cube $\hat{Q}=]a,b[$ in $\mathbb{R}^3$, let denote by $\hat{Q}'$ the associated cube $]0,b[$ and consider the family $(C'_{i,\eta,\varepsilon})_\varepsilon$. One has

$$\frac{|C_{i,\eta,\varepsilon}|}{|C'_{i,\eta,\varepsilon}|}=\frac{|C_{i,\eta,\varepsilon}|}{\left|\frac{1}{\varepsilon}Q_{i,\eta}\right|}\times\frac{|Q_{i,\eta}|}{|Q'_{i,\eta}|}\times\frac{\left|\frac{1}{\varepsilon}Q'_{i,\eta}\right|}{|C'_{i,\eta,\varepsilon}|}.$$

But one can easily check that $\lim_{\varepsilon\to 0}\frac{|C_{i,\eta,\varepsilon}|}{\left|\frac{1}{\varepsilon}Q_{i,\eta}\right|}=\lim_{\varepsilon\to 0}\frac{|C'_{i,\eta,\varepsilon}|}{\left|\frac{1}{\varepsilon}Q'_{i,\eta}\right|}=1$ so that, for ε small enough (depending on fixed η) $\frac{|C_{i,\eta,\varepsilon}|}{|C'_{i,\eta,\varepsilon}|}\le 2\frac{|Q_{i,\eta}|}{|Q'_{i,\eta}|}$. The family $(C'_{i,\eta,\varepsilon})_\varepsilon$ then clearly satisfies regularity conditions (i)-(iv).

Therefore, according to Theorem 2

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{\mathcal{S}_{C_{i,\eta,\varepsilon}}(\omega,z(x_{i,\delta,\eta}))}{|C_{i,\eta,\varepsilon}|}&=\lim_{\varepsilon\to 0}\frac{1}{|C_{i,\eta,\varepsilon}|}\int_{C_{i,\eta,\varepsilon}\setminus T(\omega)}f^{\infty,p}(\nabla w_{i,\eta,\varepsilon}(\omega,y))\,dy\\&=f_0(z_\delta(x_{i,\eta}))\end{aligned}\tag{12}$$

for all $\omega\in\Omega_{i,\eta}$ satisfying $\mathbf{P}(\Omega_{i,\eta})=1$. In what follows we denote the set of full probability $\bigcap_{\eta\in\mathbf{Q}^+}\bigcap_{i\in I_\eta}\Omega_{i,\eta}$ by Ω' and we fix $\omega\in\Omega'$. From (12) we infer

$$\begin{aligned}\int_{\mathcal{O}}f_0(z_{\delta,\eta})\,dx&=\sum_{i\in I_\eta}\int_{Q_{i,\eta}}f_0(z_{\delta,\eta})\,dx\\&=\sum_{i\in I_\eta}|Q_{i,\eta}|\,f_0(z(x_{i,\eta}))\,dx\\&=\lim_{\varepsilon\to 0}\sum_{i\in I_\eta}|Q_{i,\eta}|\,\frac{1}{|C_{i,\eta,\varepsilon}|}\int_{C_{i,\eta,\varepsilon}\setminus T(\omega)}f^{\infty,p}(\nabla w_{i,\eta,\varepsilon}(\omega,y))\,dy\\&=\lim_{\varepsilon\to 0}\sum_{i\in I_\eta}|Q_{i,\eta}|\,\frac{1}{|\varepsilon C_{i,\eta,\varepsilon}|}\int_{\varepsilon C_{i,\eta,\varepsilon}\setminus\varepsilon T(\omega)}f^{\infty,p}(\nabla w_{i,\eta,\varepsilon}(\omega,\frac{y}{\varepsilon}))dy\\&=\lim_{\varepsilon\to 0}\sum_{i\in I_\eta}\frac{\left|\frac{1}{\varepsilon}Q_{i,\eta}\right|}{|C_{i,\eta,\varepsilon}|}\int_{Q_{i,\eta}\setminus\varepsilon T(\omega)}f^{\infty,p}(\nabla w_{i,\eta,\varepsilon}(\omega,\frac{y}{\varepsilon}))dy\qquad(13)\\&=\lim_{\varepsilon\to 0}\sum_{i\in I_\eta}\int_{Q_{i,\eta}\setminus T_\varepsilon(\omega)}f^{\infty,p}(\nabla w_{i,\eta,\varepsilon}(\omega,\frac{y}{\varepsilon}))\,dy\qquad(14)\end{aligned}$$

we used the fact that $\lim_{\varepsilon\to 0}\frac{\left|\frac{1}{\varepsilon}Q_{i,\eta}\right|}{|C_{i,\eta,\varepsilon}|}=1$ and that $w_{i,\eta,\varepsilon}=0$ outside $C_{i,\varepsilon,\eta}\setminus T(\omega)$.

Let us define the function $u_{\delta,\eta,\varepsilon}$ on $\mathcal{O}$ by :

$$u_{\delta,\eta,\varepsilon}(\omega,x)=v(x)+\sum_{i\in I_\eta}w_{i,\eta,\varepsilon}(\omega,\frac{x}{\varepsilon})\mathbb{1}_{Q_{i,\eta}}(x).$$

According to the boundary condition satisfied by $w_{i,\eta,\varepsilon}$, clearly $u_{\delta,\eta,\varepsilon}\in W^{1,p}(\mathcal{O},\mathbb{R}^3)$ and $u_{\delta,\eta,\varepsilon}=v$ on $\mathcal{O}\cap T_\varepsilon(\omega)$. Furthermore, from (14), (5) and (3) we deduce

$$\begin{aligned}\int_{\mathcal{O}}f_0(z_\eta)\,dx&=\lim_{\varepsilon\to 0}\sum_{i\in I_\eta}\int_{Q_{i,\eta}\setminus T_\varepsilon(\omega)}f^{\infty,p}(\varepsilon\nabla u_{\delta,\eta,\varepsilon})\,dx\\&=\lim_{\varepsilon\to 0}\int_{\mathcal{O}\setminus T_\varepsilon(\omega)}f^{\infty,p}(\varepsilon\nabla u_{\delta,\eta,\varepsilon})\,dx\qquad(15)\\&=\lim_{\varepsilon\to 0}\varepsilon^p\int_{\mathcal{O}\setminus T_\varepsilon(\omega)}f(\nabla u_{\delta,\eta,\varepsilon})\,dx=\lim_{\varepsilon\to 0}F^v_\varepsilon(\omega,u_{\delta,\eta,\varepsilon}(\omega,.)).\qquad(16)\end{aligned}$$

Letting $\eta \to 0$, then $\delta \to 0$ in (16) and since $w \mapsto \int_{\mathcal{O}} f_0(w)\, dx$ is clearly strongly continuous in $L^p(\mathcal{O}, \mathbb{R}^3)$ we finally obtain

$$\int_{\mathcal{O}} f_0(z)\, dx = \lim_{\delta\to 0}\lim_{\eta\to 0}\lim_{\varepsilon\to 0} F_\varepsilon^v(\omega, u_{\delta,\eta,\varepsilon}(\omega,.)). \tag{17}$$

On the other hand, since $w_{i,\eta,\varepsilon} \in \mathrm{Adm}_{C_{i,\eta,\varepsilon}}(\omega, z(x_{i,\eta}))$, one has

$$\begin{aligned}
\fint_{Q_{i,\eta}} w_{i,\eta,\varepsilon}(\omega, \frac{x}{\varepsilon})\, dx &= \frac{1}{|Q_{i,\eta}|}\int_{\varepsilon C_{i,\eta,\varepsilon}} w_{i,\eta,\varepsilon}(\omega, \frac{x}{\varepsilon})\, dx \\
&= \frac{|C_{i,\eta,\varepsilon}|}{|\frac{1}{\varepsilon}Q_{i,\eta}|}\fint_{\varepsilon C_{i,\eta,\varepsilon}} w_{i,\eta,\varepsilon}(\omega, \frac{x}{\varepsilon})\, dx \\
&= \frac{|C_{i,\eta,\varepsilon}|}{|\frac{1}{\varepsilon}Q_{i,\eta}|}\fint_{C_{i,\eta,\varepsilon}} w_{i,\eta,\varepsilon}(\omega, x)\, dx \\
&= \frac{|C_{i,\eta,\varepsilon}|}{|\frac{1}{\varepsilon}Q_{i,\eta}|} z_\delta(x_{i,\eta})
\end{aligned}$$

so that letting successively $\varepsilon \to 0$ and $\eta \to 0$ we easily infer

$$\lim_{\eta\to 0}\lim_{\varepsilon\to 0} u_{\delta,\eta,\varepsilon}(\omega,.) = v + (u - v_\delta) \text{ weakly in } L^p(\mathcal{O}, \mathbb{R}^3).$$

Then letting $\delta \to 0$,

$$\lim_{\delta\to 0}\lim_{\eta\to 0}\lim_{\varepsilon\to 0} u_{\delta,\eta,\varepsilon}(\omega,.) = u \text{ weakly in } L^p(\mathcal{O}, \mathbb{R}^3).$$

A standard diagonalization argument[1] furnishes a map $\varepsilon \mapsto (\delta(\varepsilon), \eta(\varepsilon))$ such that

$$\begin{aligned}
&u_\varepsilon(\omega,.) := u_{\delta(\varepsilon),\eta(\varepsilon),\varepsilon}(\omega,.) \to u \text{ weakly converges to in } L^p(\mathcal{O}, \mathbb{R}^3); \\
&\lim_{\varepsilon\to 0} F_\varepsilon^v(\omega, u_\varepsilon(\omega,.)) = \tilde{F}_0^v(u).
\end{aligned} \tag{18}$$

Furthermore, by a standard truncation argument, we can modify the function u_ε in order that it satisfies the boundary condition $u_\varepsilon = 0$ on $\Gamma_0 := \hat{\mathcal{O}} \times \{0\}$. The conclusion of step 1 follows from (18) and (17).

Step 2. We end the proof by a relaxation argument. According to the first step $\Gamma - \limsup F_\varepsilon^v(\omega,.) \le \tilde{F}_0^v$. Taking the lower semicontinuity envelope in $L^p(\mathcal{O}, \mathbb{R}^3)$ (equipped with its weak topology) of each two member we obtain

$$\Gamma - \limsup F_\varepsilon^v(\omega,.) \le (\tilde{F}_0^v)^{**}.$$

But, from standard relaxation result $(\tilde{F}^v)^{**} = F_0^v$. □

3.2 The lower bound

Proposition 4. *For all u_ε weakly converging to u in $L^p(\mathcal{O}, \mathbb{R}^3)$ one has*

$$F_0^v(u) \le \liminf_{\varepsilon\to 0}\ F_\varepsilon^v(\omega, u_\varepsilon)$$

for $\mathbf{P}$ *a. s.* $\omega \in \Omega$.

Proof. From (3) and since $u_\varepsilon = v$ on $T_\varepsilon(\omega)$ we easily deduce

$$\varepsilon^p \liminf_{\varepsilon\to 0}\int_{\mathcal{O}\setminus T_\varepsilon} f(\nabla u_\varepsilon) dx = \liminf_{\varepsilon\to 0}\int_{\mathcal{O}} f^{\infty,p}(\varepsilon\nabla u_\varepsilon) dx.$$

Fix x_0 in $\mathcal{O}$ and set $Q_\rho(x_0) := S_\rho(\hat{x}_0) \times I_\rho(x_{0,3})$ (to shorten notation we sometimes do not indicate the fixed argument x_0). By using a blow up argument, it is enough to prove that for a.e. x_0 one has

$$\lim_{\rho\to 0}\liminf_{\varepsilon\to 0}\fint_{Q_\rho(x_0)} f^{\infty,p}(\varepsilon\nabla u_\varepsilon)\, dx \ge f_0^{**}(u(x_0) - v(x_0)).$$

[1]One can easily check that $u_{\delta,\eta,\varepsilon}(\omega,.)$ belongs to a fixed ball $\mathcal{B}(0,r)$ of $L^p(\mathcal{O}, \mathbb{R}^3)$. Since the weak topology of $L^p(\mathcal{O}, \mathbb{R}^3)$ induces a metric on bounded sets, the diagonalization argument holds.

According to the decomposition lemma (cf [3, 11]), there exists w_ε in $W_0^{1,p}(Q_\rho,\mathbb{R}^3)$ such that $(|\nabla w_\varepsilon|^p)_{\varepsilon>0}$ is uniformly integrable and such that the sequences $(\nabla w_\varepsilon)_{\varepsilon>0}$ and $(\nabla\varepsilon u_\varepsilon)_{\varepsilon>0}$ generate the same Young measure μ (for shorten notation we do note indicate the dependance on ρ for w). Therefore applying standard lower semicontinuity and continuity properties for Young measures (see Proposition 4.3.4 and Theorem 4.3.3 in [3]) we infer

$$\liminf_{\varepsilon\to 0} \fint_{Q_\rho(x_0)} f^{\infty,p}(\nabla w_\varepsilon)\,dx = \int_{Q_\rho(x_0)}\int_{\mathbf{M}_S^{3\times 3}} f^{\infty,p}(M)\,d\mu_x\,dx \le \liminf_{\varepsilon\to 0}\fint_{Q_\rho(x_0)} f^{\infty,p}(\varepsilon\nabla u_\varepsilon)\,dx$$

so that it suffices to establish

$$\liminf_{\rho\to 0}\liminf_{\varepsilon\to 0}\fint_{Q_\rho(x_0)} f^{\infty,p}(\nabla w_\varepsilon)\,dx \ge f_0^{**}(u(x_0)-v(x_0)). \tag{19}$$

Note that since

$$\begin{aligned}&\varepsilon u_\varepsilon \rightharpoonup 0 \text{ in } W^{1,p}(\mathcal{O},\mathbb{R}^3)\\ &\varepsilon u_\varepsilon \to 0 \text{ in } L^p(\mathcal{O},\mathbb{R}^3)\end{aligned}$$

we easily deduce

$$w_\varepsilon \rightharpoonup 0 \text{ in } W^{1,p}(Q_\rho,\mathbb{R}^3) \tag{20}$$

$$w_\varepsilon \to 0 \text{ in } L^p(Q_\rho,\mathbb{R}^3) \tag{21}$$

when $\varepsilon\to 0$.

Let $C_{\varepsilon,\rho}$ be the smaller cube in $\mathcal{I}$ containing $\frac{1}{\varepsilon}Q_\rho$. Our strategy consists in modifying the function w_ε in order to obtain a function in $\mathrm{Adm}_{C_{\varepsilon,\rho}}(\omega, u(x_0)-v(x_0))$ whose gradient decreases the left hand side of (19). For this we will make various changes on w_ε.

First change. For $\eta>0$ intended to go to 0 set $\hat{A}_\eta := (S_\rho\setminus\varepsilon D(\omega))_\eta$ (we do not indicate the dependence on ε and ω) and $w_{\varepsilon,\eta}(x) := \phi_{\varepsilon,\eta}(\hat{x})w_\varepsilon(x)+\varepsilon v(x_0)$ where $\phi_{\varepsilon,\eta} := \rho_\eta * 1_{\hat{A}_\eta}$. Note that $\phi_{\varepsilon,\eta}$ satisfies

$$|grad(\phi_{\varepsilon,\eta})| \le \frac{C}{\eta}$$

and

$$\phi_{\varepsilon,\eta} = \begin{cases} 0 \text{ in } \partial(S_\rho\setminus\varepsilon D(\omega)),\\ 1 \text{ in } \hat{A}_{2\eta},\end{cases}$$

thus $w_{\varepsilon,\eta} := \varepsilon v(x_0)$ on $\partial(Q_\rho\setminus\varepsilon T(\omega))$. We extend $w_{\varepsilon,\eta}$ by $\varepsilon v(x_0)$ on the complementary set of $Q_\rho\setminus\varepsilon T(\omega)$. Set $A_{2\eta} := \hat{A}_{2\eta}\times I_\rho$. From the growth condition (3) we deduce

$$\begin{aligned}&\int_{Q_\rho} f^{\infty,p}(\nabla w_{\varepsilon,\eta})\,dx\\ =\;& \int_{A_{2\eta}} f^{\infty,p}(\nabla w_{\varepsilon,\eta})\,dx + \int_{(Q_\rho\setminus\varepsilon T(\omega))\setminus A_{2\eta}} f^{\infty,p}(\nabla w_{\varepsilon,\eta})\,dx\\ \le\;& \int_{Q_\rho} f^{\infty,p}(\nabla w_\varepsilon)\,dx + \int_{(Q_\rho\setminus\varepsilon T(\omega))\setminus A_{2\eta}} f^{\infty,p}(\nabla w_{\varepsilon,\eta})\,dx.\end{aligned} \tag{22}$$

On the other hand

$$\begin{aligned}&\int_{(Q_\rho\setminus\varepsilon T(\omega))\setminus A_{2\eta}} f^{\infty,p}(\nabla w_{\varepsilon,\eta})\,dx\\ \le\;& C\int_{(Q_\rho\setminus\varepsilon T(\omega))\setminus A_{2\eta}} \big(||grad\ \phi||^p|w_\varepsilon|^p + |\phi_{\varepsilon,\eta}|^p|\nabla w_\varepsilon|^p\big)\,dx\\ \le\;& C\Big(\frac{1}{\eta^p}\int_{(Q_\rho\setminus\varepsilon T(\omega))\setminus A_{2\eta}}|w_\varepsilon|^p dx + \sup_\varepsilon\int_{(Q_\rho\setminus\varepsilon T(\omega))\setminus A_{2\eta}}|\nabla w_\varepsilon|^p dx\Big).\end{aligned}$$

Letting successively $\varepsilon \to 0$ and $\eta \to 0$ and since $(|\nabla w_\varepsilon|^p)_{\varepsilon>0}$ is uniformly integrable we finally deduce

$$\limsup_{\eta} \limsup_{\varepsilon} \int_{(Q_\rho \setminus \varepsilon T(\omega)) \setminus A_{2\eta}} f^{\infty,p}(\nabla w_{\varepsilon,\eta})\ dx = 0. \tag{23}$$

Consequently, combining (22) and (23)

$$\liminf_{\eta} \liminf_{\varepsilon} \int_{Q_\rho} f^{\infty,p}(\nabla w_{\varepsilon,\eta})\ dx \quad \leq \quad \liminf_{\varepsilon} \fint_{Q_\rho} f^{\infty,p}(\nabla w_\varepsilon)\ dx. \tag{24}$$

Second change. With the function $\phi_{\varepsilon,\eta}$ defined previously, set $\psi_{\varepsilon,\eta} := \frac{\phi_{\varepsilon,\eta}}{\fint_{S_\rho} \phi_{\varepsilon,\eta}\ d\hat{x}}$ and consider the random function

$$z_{\varepsilon,\eta} := w_{\varepsilon,\eta} + \psi_{\varepsilon,\eta}[\varepsilon u(x_0) - \fint_{Q_\rho} w_{\varepsilon,\eta}\ dx].$$

The function $\psi_{\varepsilon,\eta}$ fulfills the following conditions

$$\psi_{\varepsilon,\eta}(\hat{x}) \quad = \quad \begin{cases} 0 \quad \text{on } \partial(S_\rho \setminus \varepsilon D(\omega)), \\ \dfrac{1}{\fint_{S_\rho} \phi_{\varepsilon,\eta}\ d\hat{x}} \quad \text{on } \hat{A}_{2\eta}, \end{cases}$$

$\fint_{S_\rho} \psi_{\varepsilon,\rho}\ d\hat{x} = 1$, and $|grad(\psi_{\varepsilon,\eta})| \leq \dfrac{1}{C(\eta)\eta}$ where $C(\eta)$ is a nonnegative constants depending only of ρ and η (change of scale, reason on $\frac{1}{\varepsilon} S_\rho \setminus D(\omega)$ and argue as in (7)). Thus

$$\begin{cases} \displaystyle\fint_{Q_\rho} z_{\varepsilon,\eta}\ dx = \varepsilon u(x_0) \\ \text{and} \\ z_{\varepsilon,\eta} = \varepsilon v(x_0) \text{ on } \partial(Q_\rho \setminus \varepsilon T(\omega)). \end{cases}$$

From the definition of $z_{\varepsilon,\eta}$ we derive

$$\begin{aligned} \int_{Q_\rho} f^{\infty,p}(\nabla z_{\varepsilon,\eta})\ dx \quad &= \quad \int_{A_{2\eta}} f^{\infty,p}(\nabla w_{\varepsilon,\eta})\ dx + R_{\varepsilon,\eta,\rho} \\ &\leq \quad \int_{Q_\rho} f^{\infty,p}(\nabla w_{\varepsilon,\eta})\ dxdx + R_{\varepsilon,\eta,\rho} \end{aligned} \tag{25}$$

where

$$R_{\varepsilon,\eta,\rho} := \int_{(Q_\rho \setminus \varepsilon T(\omega)) \setminus A_{2\eta}} f^{\infty,p}(\nabla z_{\varepsilon,\eta})\ dx.$$

From the growth condition (4)

$$|R_{\varepsilon,\eta,\rho}| \leq \int_{(Q_\rho \setminus \varepsilon T(\omega)) \setminus A_{2\eta}} |\nabla w_{\varepsilon,\eta}|^p\ \ dx + C\frac{|(Q_\rho \setminus \varepsilon T(\omega)) \setminus A_{2\eta}|^p}{\eta^p C(\eta)^p} \Big[\varepsilon^p |u(x_0)|^p + \Big|\fint_{Q_\rho} w_{\varepsilon,\eta}\ d\hat{x}\Big|^p\Big].$$

But applying estimate (23) with the function $|\ .\ |^p$ substituted for $f^{\infty,p}$

$$\limsup_{\eta\to 0} \limsup_{\varepsilon\to 0} \int_{(Q_\rho \setminus \varepsilon T(\omega)) \setminus A_{2\eta}} |\nabla w_{\varepsilon,\eta}|^p\ \ dx = 0.$$

On the other hand

$$\Big[\varepsilon^p |u(x_0)|^p + \Big|\fint_{Q_\rho} w_{\varepsilon,\eta}\ d\hat{x}\Big|^p\ \Big]$$

clearly tends to 0 when $\varepsilon \to 0$. Thus, letting successively $\varepsilon \to 0$ and $\eta \to 0$ in (25) we obtain

$$\liminf_{\eta\to 0}\liminf_{\varepsilon\to 0}\int_{Q_\rho} f^{\infty,p}(\nabla z_{\varepsilon,\eta})\,dx \le \liminf_{\eta\to 0}\liminf_{\varepsilon\to 0}\int_{Q_\rho} f^{\infty,p}(\nabla w_{\varepsilon,\eta})\,dx$$

and finally from (24)

$$\liminf_{\eta\to 0}\liminf_{\varepsilon\to 0}\int_{Q_\rho} f^{\infty,p}(\nabla z_{\varepsilon,\eta})\,dx \le \liminf_{\varepsilon} \fint_{Q_\rho} f^{\infty,p}(\nabla w_\varepsilon)\,dx. \tag{26}$$

Let us define the Sobolev function $z'_{\varepsilon,\eta}$ on $\frac{1}{\varepsilon}Q_\rho$ by $z'_{\varepsilon,\eta}(y) := \dfrac{1}{\varepsilon} z_{\varepsilon,\eta}(\varepsilon y)$. It satisfies the following condition

$$\begin{aligned}
z'_{\varepsilon,\eta}(y) &= v(x_0) \quad \text{on } \partial(\frac{1}{\varepsilon}Q_\rho\backslash T(\omega))\\
\fint_{Q_\rho} z'_{\varepsilon,\eta}(y)dy &= \frac{1}{\varepsilon}\fint_{\frac{1}{\varepsilon}Q_\rho} z_{\varepsilon,\eta}(\varepsilon y)dy\\
&= \frac{1}{\varepsilon}\fint_{Q_\rho} z_{\varepsilon,\eta}(x)dx\\
&= u(x_0).
\end{aligned}$$

Extend $z'_{\varepsilon,\eta}$ by $v(x_0)$ on $C_{\varepsilon,\rho}\setminus \frac{1}{\varepsilon}Q_\rho$ where $C_{\varepsilon,\rho}$ denotes the smaller cube in $\mathcal{I}$ containing $\frac{1}{\varepsilon}Q_\rho$, and set

$$z"_{\varepsilon,\eta} := \frac{|C_{\varepsilon,\eta}|}{|\frac{1}{\varepsilon}S_\rho|}\big(z'_{\varepsilon,\eta} - v(x_0)\big).$$

It is easy to check that $z"_{\varepsilon,\eta}$ belongs to $\mathrm{Adm}_{C_{\varepsilon,\eta}}(u(x_0)-v(x_0))$. Moreover, a change of scale at the left hand side of (26), the facts that $f^{\infty,p}$ is positively p-homogeneous and $\lim_{\varepsilon\to 0}\dfrac{|C_{\varepsilon,\eta}|}{|\frac{1}{\varepsilon}Q_\rho|} = 1$ yield for $\mathbf{P}$ almost every ω

$$\begin{aligned}
f_0(u(x_0)-v(x_0)) &= \lim_{\varepsilon\to 0}\frac{1}{|C_{\varepsilon,\rho}|}\mathcal{S}_{C_{\varepsilon,\rho}}(\omega, u(x_0)-v(x_0))\\
&\le \limsup_{\eta\to 0}\limsup_{\varepsilon\to 0}\fint_{C_{\varepsilon,\rho}} f^{\infty,p}(\nabla z"_{\varepsilon,\eta})\,dx\\
&= \limsup_{\eta\to 0}\limsup_{\varepsilon\to 0}\fint_{\frac{1}{\varepsilon}Q_\rho} f^{\infty,p}(\nabla z'_{\varepsilon,\eta})\,dx\\
&= \liminf_{\eta\to 0}\liminf_{\varepsilon\to 0}\fint_{Q_\rho} f^{\infty,p}(\nabla z_{\varepsilon,\eta})\,dx\\
&\le \liminf_{\varepsilon}\fint_{Q_\rho} f^{\infty,p}(\nabla w_\varepsilon)\,dx.
\end{aligned}$$

Therefore $f_0^{**}(u(x_0)-v(x_0)) \le \liminf_{\varepsilon}\fint_{Q_\rho} f^{\infty,p}(\nabla w_\varepsilon)\,dx$ which completes the proof. □

Remark 3. *A carefully analysis of the proof above lead us to the following generalization of the lover bound: for all $u_\varepsilon \rightharpoonup u$ in $L^p(\mathcal{O},\mathbb{R}^3)$ for all function v_ε satisfying $\sup_{\varepsilon>0}\|\nabla v_\varepsilon)\|_{L^p(\mathcal{O}\cap T_\varepsilon,\mathbf{M}^{3\times 3}_S)} < +\infty$ and for all function $\zeta \in L^p(\mathcal{O},\mathbb{R}^3)$*

$$F^\zeta(u) \le \liminf_{\varepsilon\to 0} F_\varepsilon^{v_\varepsilon}(\omega, u_\varepsilon)$$

where, for every $u \in L^p(\mathcal{O},\mathbb{R}^3)$,

$$F_\varepsilon^{v_\varepsilon}(\omega,u) = \begin{cases} \displaystyle\int_{\mathcal{O}\setminus T_\varepsilon} f(\varepsilon\nabla u)\ dx \ \textit{if}\ u \in W^{1,p}_{\Gamma_0}(\mathcal{O},\mathbb{R}^3), u = v_\varepsilon \ \textit{on}\ \mathcal{O}\cap T_\varepsilon \\ +\infty \ \textit{otherwise,} \end{cases}$$

$$F^\zeta(u) = \int_{\mathcal{O}} f_0^{**}(u-\zeta).$$

4 The limit problem associated with the fibers

4.1 The limit functional

In the following we denote by $a(\omega,.)$ the characteristic function of the random set $D(\omega)$ so that $1_{D_\varepsilon}(\omega)(\hat{x}) = 1_{D_r(\omega)}(\frac{\hat{x}}{\varepsilon}) := a(\omega,\frac{\hat{x}}{\varepsilon})\ \ \forall \hat{x} \in \widehat{\mathcal{O}}$. With this notation we can rewrite the functional

$$G_\varepsilon(\omega,u):\ L^p(\mathcal{O},\mathbb{R}^3) \to \mathbb{R}\cup\{\infty\}$$

as follows:

$$G_\varepsilon(\omega,u) = \begin{cases} \displaystyle\int_{\mathcal{O}} a(\omega,\frac{\hat{x}}{\varepsilon}) g(\nabla_\varepsilon u) dx \text{ if } u\lfloor(\mathcal{O}\setminus T_\varepsilon) = 0,\ u\lfloor(\mathcal{O}\cap T_\varepsilon) \in W^{1,p}_{\Gamma_0}(\mathcal{O}\cap T_\varepsilon,\mathbf{R}^3) \\ +\infty \text{ otherwise.} \end{cases}$$

According to Proposition 2 of Section 2 we infer

Proposition 5. *There exists a* $\mathbf{P}$*-mesurable set* $\Omega" \subset \Omega$*, with* $P(\Omega") = 1$ *such that*

$$\forall \omega \in \Omega",\quad a(\omega,\frac{\cdot}{\varepsilon}) \rightharpoonup \mathbf{E}(\int_{\hat{Y}} a(\omega,\hat{y})d\hat{y}) := \mathbb{E} \quad \textit{for the topology}\ \ \sigma(L^\infty,L^1).$$

Now let us consider the function $g^\perp : \mathbb{R}^3 \to \mathbb{R}$ defined for every $a \in \mathbb{R}$ by

$$g^\perp(a) := \inf_{\xi\in\mathbf{M}^{3\times 2}} g\left(\xi|a\right),$$

then we define the deterministic functional $G_0 : L^p(\mathcal{O},\mathbb{R}^3) \to \mathbb{R}\cup\{\infty\}$ by

$$G_0(u) = \begin{cases} \mathbb{E}\displaystyle\int_{\mathcal{O}} (g^\perp)^{**}(\frac{1}{\mathbb{E}}\frac{\partial u}{\partial x_3})dx \text{ if } \ u \in V_0 \\ +\infty \text{ otherwise} \end{cases}$$

where $V_0 := \left\{ u \in L^p(\mathcal{O},\mathbb{R}^3) : \dfrac{\partial u}{\partial x_3} \in L^p(\mathcal{O},\mathbb{R}^3),\ u(\hat{x},0) = 0 \text{ on } \widehat{\mathcal{O}} \right\}$. In the next sections we are going to establish

Theorem 4. *The sequence* $(G_\varepsilon)_{\varepsilon>0}$ *almost surely sequentially* Γ*-converges to the functional* G_0 *when* $L^p(\mathcal{O},\mathbb{R}^3)$ *is equipped with its weak topology.*

The use of the weak topology in $L^p(\mathcal{O},\mathbb{R}^3)$ comes from the next compactness result.

Lemma 2. *Let* $(u_\varepsilon)_{\varepsilon>0}$ *be a sequence satisfying* $\sup_\varepsilon G_\varepsilon(\omega,u_\varepsilon) < +\infty$ *for all* $\omega \in \Omega"$*. Then for all* $\omega \in \Omega"$ *there exist a subsequence possibly depending on* ω *and* $u \in V_0$ *possibly depending on* ω *such that*

$$a(\omega,\frac{\cdot}{\varepsilon})u_\varepsilon \rightharpoonup u \ \textit{in}\ L^p(\mathcal{O},\mathbb{R}^3) \tag{27}$$

$$a(\omega,\frac{\cdot}{\varepsilon})\frac{\partial u_\varepsilon}{\partial x_3} \rightharpoonup \frac{\partial u}{\partial x_3}\ \textit{in}\ L^p(\mathcal{O},\mathbb{R}^3). \tag{28}$$

Proof. From the coercivity of g, we infer

$$\int_{\mathcal{O}\cap T_\varepsilon} |u_\varepsilon(\hat{x},x_3)|^p dx \leq C\int_{\mathcal{O}\cap T_\varepsilon} \left|\frac{\partial u_\varepsilon}{\partial x_3}\right|^p dx \leq \frac{C(\omega)}{\alpha} G_\varepsilon(\omega,u_\varepsilon)$$

which gives (27). Weak convergence (28) is obvious and $u(\hat{x},0)=0$ on $\widehat{\mathcal{O}}$ is easily checked. Note that since $V_0 \subset W^{1,p}((0,h),L^p(\mathcal{O},\mathbb{R}^3)) \subset \mathcal{C}([0,h],L^p(\mathcal{O},\mathbb{R}^3))$ equality $u(.,0)=0$ may be understood in a classical sense. □

4.2 The lower bound

Proposition 6. *For all u_ε such that $a(\omega,\frac{\cdot}{\varepsilon})u_\varepsilon$ weakly converges to u in $L^p(\mathcal{O},\mathbb{R}^3)$ and all $\omega\in\Omega$"*

$$G_0(u) \leq \liminf_{\varepsilon\to 0} G_\varepsilon(\omega,u_\varepsilon)$$

Proof. Fix $\omega\in\Omega$" and assume that $\liminf_{\varepsilon\to0} G_\varepsilon(u_\varepsilon) < +\infty$. From inequalities $g\geq g^\perp \geq (g^\perp)^{**}$ and the the Moreau-Rockafellar duality principle we infer that for all ϕ in $L^q(\mathcal{O},\mathbb{R}^3)$ where $q=\frac{p}{p-1}$ is the conjugate exponent of p:

$$\begin{aligned}\liminf_{\varepsilon\to0} G_\varepsilon(u_\varepsilon) &\geq \liminf_{\varepsilon\to0}\int_{\mathcal{O}} a(\omega,\frac{\hat{x}}{\varepsilon})(g^\perp)^{**}(\frac{\partial u_\varepsilon}{\partial x_3})dx\\ &\geq \liminf_{\varepsilon\to0}\Big(\int_{\mathcal{O}} a(\omega,\frac{\hat{x}}{\varepsilon})\phi.\frac{\partial u_\varepsilon}{\partial x_3}dx - \int_{\mathcal{O}} a(\omega,\frac{\hat{x}}{\varepsilon})(g^\perp)^*(\phi)dx\Big)\\ &= \int_{\mathcal{O}}\phi.\frac{\partial u}{\partial x_3}dx - \mathbb{E}\int_{\mathcal{O}}(g^\perp)^*(\phi)dx\\ &= \mathbb{E}\left[\int_{\mathcal{O}}\frac{1}{\mathbb{E}}\phi\frac{\partial u}{\partial x_3}dx - \int_{\mathcal{O}}(g^\perp)^*(\phi)dx\right].\end{aligned}$$

By taking the the supremum over all functions ϕ in $\phi\in L^q(\mathcal{O},\mathbb{R}^3)$ we finally obtain

$$\begin{aligned}\liminf_{\varepsilon\to0} G_\varepsilon(u_\varepsilon) &\geq \mathbb{E}\sup_{\phi\in L^q(\mathcal{O},\mathbb{R}^3)}\left[\int_{\mathcal{O}}\frac{1}{\mathbb{E}}\phi\frac{\partial u}{\partial x_3}dx - \int_{\mathcal{O}}(g^\perp)^*(\phi)dx\right]\\ &= \mathbb{E}\int_{\mathcal{O}}(g^\perp)^{**}(\frac{1}{\mathbb{E}}\frac{\partial u}{\partial x_3})dx\end{aligned}$$

which completes the proof. □

4.3 The upper bound

Proposition 7. *For all $u\in V_0$ and all $\omega\in\Omega$" there exists a sequence $(u_\varepsilon(\omega))_{\varepsilon>0}$ in $L^p(\mathcal{O},\mathbb{R}^3)$ weakly converging to u in $L^p(\mathcal{O},\mathbb{R}^3)$ such that*

$$G_0(u) \geq \limsup_{\varepsilon\to0} G_\varepsilon(\omega,u_\varepsilon(\omega)).$$

Proof. In all the proof we fix ω in Ω". We proceed into two steps.

First step. let $u\in C^1(\overline{\mathcal{O}},\mathbb{R}^3)$, with $u=0$ on Γ_0. We construct a sequence $(u_\varepsilon(\omega))_{\varepsilon>0}\in L^p(\mathcal{O},\mathbb{R}^3)$ such that $u_\varepsilon\lfloor(\mathcal{O}\backslash T_\varepsilon)=0$, $u_\varepsilon\lfloor(\mathcal{O}\cap T_\varepsilon)\in W^{1,p}(\mathcal{O}\cap T_\varepsilon,\mathbb{R}^3)$ and satisfying

$$\begin{aligned}&a(\omega,\frac{\cdot}{\varepsilon})u_\varepsilon(\omega) \rightharpoonup u \text{ in } L^p(\mathcal{O},\mathbb{R}^3),\\ &\lim_{\varepsilon\to0} G_\varepsilon(\omega,u_\varepsilon(\omega)) = \mathbb{E}\int_{\mathcal{O}} g^\perp(\frac{1}{\mathbb{E}}\frac{\partial u}{\partial x_3})dx.\end{aligned}$$

Consider (ξ^η) in $L^p(\mathcal{O}, \mathbf{M}^{3\times 2})$ such that

$$\begin{aligned}\mathbb{E}\int_{\mathcal{O}} g^\perp(\frac{1}{\mathbb{E}}\frac{\partial u}{\partial x_3})dx &= \mathbb{E}\int_{\mathcal{O}} \inf_{\xi\in\mathbf{M}^{3\times 2}} g\left(\xi + \frac{1}{\mathbb{E}}\hat{\nabla} u,\ \frac{1}{\mathbb{E}}\frac{\partial u}{\partial x_3}\right)dx \\ &\geq \mathbb{E}\int_{\mathcal{O}} g\left(\xi^\eta + \hat{\nabla} u, \frac{1}{\mathbb{E}}\frac{\partial u}{\partial x_3}\right)dx - \eta. \end{aligned} \tag{29}$$

The measurability of the matrix valued function $x \mapsto \xi^\eta(x)$ may be proven thanks to the measurable selection theorem (see [7]). Its summability comes from the coercivity and the growth condition fulfilled by g. Since $C^1_c(\mathcal{O}, \mathbf{M}^{3\times 2})$ is dense in $L^p(\mathcal{O}, \mathbf{M}^{3\times 2})$, according to the Lipschitz property of the convex function g one may assume that $(\xi^\eta) \in C^1_c(\mathcal{O}, \mathbf{M}^{3\times 2})$.

Let us consider a random function $\phi(\omega, .) = (\phi_1(\omega, .), \phi_2(\omega, .))$ in $\mathcal{C}_c(\mathbb{R}^2, \mathbb{R}^2)$ satisfying $\phi(\omega, \hat{y}) = \hat{y} - \omega$ whenever $\hat{y} \in \omega + K_i$ and set

$$u_{\varepsilon,\eta} = a(\omega, \frac{\hat{x}}{\varepsilon})\big[\frac{1}{\mathbb{E}}u(x) + \varepsilon\phi_1(\omega, \frac{\widehat{x}}{\varepsilon})\xi_1^\eta + \varepsilon\phi_2(\omega, \frac{\widehat{x}}{\varepsilon})\xi_2^\eta\big]. \tag{30}$$

Clearly $u_{\varepsilon,\eta} \in W^{1,p}_{\Gamma_0}(\mathcal{O}\cap T_\varepsilon, \mathbb{R}^3)$ and $u_{\varepsilon,\eta}\lfloor(\mathcal{O}\backslash T_\varepsilon) = 0$. Furthermore, from Proposition 5, $u_{\varepsilon,\eta}(\omega) \rightharpoonup u$ in $L^p(\mathcal{O}, \mathbb{R}^3)$. On the other hand a straightforward calculation yields

$$\begin{aligned}\hat{\nabla} u_{\varepsilon,\eta} &= \tfrac{1}{\mathbb{E}}\hat{\nabla} u + \xi^\eta + O_\eta(\varepsilon),\\ \tfrac{\partial u_{\varepsilon,\eta}}{\partial x_3} &= \tfrac{1}{\mathbb{E}}\tfrac{\partial u}{\partial x_3} + O_\eta(\varepsilon)\end{aligned}$$

on $T_\varepsilon \cap \mathcal{O}$, where $\lim_{\varepsilon\to 0} O_\eta(\varepsilon) = 0$. From (29) and the Lebesgue dominated convergence theorem we infer

$$\begin{aligned}\lim_{\varepsilon\to 0} G_\varepsilon(u_{\varepsilon,\eta}(\omega)) &= \mathbb{E}\int_{\mathcal{O}} g\left(\xi^\eta + \frac{1}{\mathbb{E}}\hat{\nabla} u,\ \frac{1}{\mathbb{E}}\frac{\partial u}{\partial x_3}\right)dx \\ &\leq \mathbb{E}\int_{\mathcal{O}}(g^\perp)(\frac{1}{\mathbb{E}}\frac{\partial u}{\partial x_3})dx + \eta.\end{aligned}$$

By using a standard diagonalization argument, there exists a map $\varepsilon \mapsto \eta(\varepsilon)$, $\eta(\varepsilon) \to 0$ when $\varepsilon \to 0$ so that, setting $u_\varepsilon = u_{\varepsilon,\eta(\varepsilon)}$

$$\begin{cases} u_\varepsilon(\omega) \rightharpoonup u \quad \text{ in } L^p(\mathcal{O}, \mathbb{R}^3) \\ \lim\limits_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon(\omega)) \leq \mathbb{E}\displaystyle\int_{\mathcal{O}}(g^\perp)(\frac{1}{\mathbb{E}}\frac{\partial u}{\partial x_3})dx. \end{cases}$$

Second step (Relaxation). let $u \in V_0$. Thus $G(u) = \mathbb{E}\int_{\mathcal{O}}(g^\perp)^{**}(\frac{1}{\mathbb{E}}\frac{\partial u}{\partial x_3})\ dx$. According to standard relaxation results there exists a sequence $(\theta_n)_{n\in\mathbb{N}}$ in $\mathcal{C}^1_c(\mathcal{O}, \mathbb{R}^3)$ weakly converging to $\frac{\partial u}{\partial x_3}$ in $L^p(\mathcal{O}, \mathbb{R}^3)$ such that

$$\lim_{n\to+\infty} \int_{\mathcal{O}} g^\perp(\frac{1}{\mathbb{E}}\theta_n) = \int_{\mathcal{O}}(g^\perp)^{**}(\frac{1}{\mathbb{E}}\frac{\partial u}{\partial x_3})dx. \tag{31}$$

For all $x \in \mathcal{O}$ set

$$v_n(x) := \int_0^{x_3} \theta_n(\hat{x}, s)\ ds \quad \text{with } v_n \in V_0.$$

Thus $\frac{\partial v_n}{\partial x_3} \rightharpoonup \frac{\partial u}{\partial x_3}$ in $L^p(\mathcal{O}, \mathbb{R}^3)$ so that $v_n \rightharpoonup u$ in $L^p(\mathcal{O}, \mathbb{R}^3)$. From (31) we infer that $(v_n)_{n\in\mathbb{N}}$ is a sequence of $\mathcal{C}^1(\overline{\mathcal{O}}, \mathbb{R}^3)$-functions in V_0 weakly converging to u in $L^p(\mathcal{O}, \mathbb{R}^3)$ and satisfying

$$\lim_{n\to+\infty} \mathbb{E}\int_{\mathcal{O}} g^\perp(\frac{1}{\mathbb{E}}\frac{\partial v_n}{\partial x_3}) = G(u).$$

Last step. With the notation of the previous step, according to the first step there exists a sequence $(u_{\varepsilon,n}(\omega))_{\varepsilon>0}$ satisfying

$$\begin{cases} u_{\varepsilon,n}(\omega) \rightharpoonup v_n & \text{in } L^p(\mathcal{O},\mathbb{R}^3) \text{ when } \varepsilon \to 0, \\ \lim_{\varepsilon\to 0} G_{\varepsilon,n}(u_{\varepsilon,n}(\omega)) \leq \mathbb{E}\int_{\mathcal{O}} (g^{\perp})(\frac{1}{\mathbb{E}}\frac{\partial v_n}{\partial x_3})dx. \end{cases}$$

Letting $n \to +\infty$ in the two estimates above and using again a standard diagonalization argument, we deduce that there exists a map $\varepsilon \mapsto n(\varepsilon)$ such that

$$\begin{cases} u_{\varepsilon,n(\varepsilon)}(\omega) \rightharpoonup u & \text{in } L^p(\mathcal{O},\mathbb{R}^3) \\ \lim_{\varepsilon\to 0} G_{\varepsilon}(\omega, u_{\varepsilon,n(\varepsilon)}(\omega)) \leq \mathbb{E}\int_{\mathcal{O}} (g^{\perp})^{**}(\frac{1}{\mathbb{E}}\frac{\partial u}{\partial x_3})dx. \end{cases} \tag{32}$$

We end the proof by setting $u_\varepsilon(\omega) := u_{\varepsilon,n(\varepsilon)}(\omega)$. □

5 The limit problem associated with the complete structure

Now, we deal with the asymptotic behavior of the complete structure. Let us recall that the functional energy H_ε is defined on $L^p(\mathcal{O},\mathbb{R}^3)$ by :

$$H_\varepsilon(\omega,u) = \begin{cases} \int_{\mathcal{O}\setminus T_\varepsilon} \varepsilon^p f(\nabla u)\, dx + \int_{\mathcal{O}\cap T_\varepsilon} g(\nabla_\varepsilon u)\, dx \text{ if } u \in W^{1,p}_{\Gamma_0}(\mathcal{O},\mathbb{R}^3) \\ +\infty \text{ otherwise.} \end{cases}$$

It is worth noticing that for u in $W^{1,p}_{\Gamma_0}(\mathcal{O},\mathbb{R}^3)$, one has

$$H_\varepsilon(\omega,u) = F^u_\varepsilon(\omega,u) + G_\varepsilon(\omega, 1_{T_\varepsilon\cap\mathcal{O}}u).$$

We equip $L^p(\mathcal{O},\mathbb{R}^3)$ with its weak topology. The main theorem of the paper is

Theorem 5. *The sequence $(H_\varepsilon)_{\varepsilon>0}$ almost surely sequentially Γ-converges to the infimum convolution $F_0 \bigtriangledown G_0$ defined for every $u \in L^p(\mathcal{O},\mathbb{R}^3)$ by*

$$F_0 \bigtriangledown G_0\ (u) := \inf_{v\in L^p(\mathcal{O},\mathbb{R}^3)} \Big(F_0(u-v) + G_0(\mathbb{E}v)\Big) = \inf_{v\in L^p(\mathcal{O},\mathbb{R}^3)} \Big(F^v_0(u) + G_0(\mathbb{E}v)\Big).$$

Consequently $(H_\varepsilon + \mathcal{L})_{\varepsilon>0}$ almost surely sequentially Γ-converges to the functional $F_0 \bigtriangledown G_0 + \mathcal{L}$.

The choice of the weak topology which equips $L^p(\mathcal{O},\mathbb{R}^3)$ is suggested by the following compactness result. The proof is very similar to that of Lemma 1 and 2 and left to the reader.

Lemma 3. *Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence in $L^p(\mathcal{O},\mathbb{R}^3)$ such that $\sup_{\varepsilon>0} H_\varepsilon(\omega,u_\varepsilon) < +\infty$ and set $v_\varepsilon = a(\omega,\frac{\cdot}{\varepsilon})u_\varepsilon$. Then, there exist $(u,\mathbb{E}v)$ in $L^p(\mathcal{O},\mathbb{R}^3)$ and a subsequence possibly depending on ω such that for $\mathbf{P}$ almost every ω*

$$(u_\varepsilon, v_\varepsilon) \rightharpoonup (u,\mathbb{E}v) \text{ in } L^p(\mathcal{O},\mathbb{R}^3)\times L^p(\mathcal{O},\mathbb{R}^3)$$

$$\frac{\partial v_\varepsilon}{\partial x_3} \rightharpoonup \frac{\partial \mathbb{E}v}{\partial x_3} \text{ in } L^p(\mathcal{O},\mathbb{R}^3).$$

5.1 The lower bound

In this section, we establish the lower bound in the definition of the Γ-convergence of H_ε to H:

Proposition 8. *For every u_ε weakly converging to u in $L^p(\mathcal{O},\mathbb{R}^3)$, and for $\mathbf{P}$-almost every ω in Ω*

$$\mathcal{H}(u) \leq \liminf_{\varepsilon\to 0} H_\varepsilon(\omega,u_\varepsilon).$$

Proof. One may assume $\liminf_{\varepsilon\to 0} H_\varepsilon(\omega,u_\varepsilon)<+\infty$, so that, for a non relabeled subsequence,

$$H_\varepsilon(\omega,u)=F_\varepsilon^{u_\varepsilon}(\omega,u_\varepsilon)+G_\varepsilon(\omega,1_{T_\varepsilon\cap\mathcal{O}}u_\varepsilon).$$

According to Proposition 6 and Remark 3 we obtain for all $v\in L^p(\mathcal{O},\mathbb{R}^3)$ and for $\mathbf{P}$ a.s. $\omega\in\Omega$

$$\liminf_{\varepsilon\to 0} H_\varepsilon(\omega,u_\varepsilon)\geq F_0^v(u)+G_0(\mathbb{E}v),$$

thus

$$\liminf_{\varepsilon\to 0} H_\varepsilon(\omega,u_\varepsilon)\geq \inf_{\theta\in L^p(\mathcal{O},\mathbb{R}^3)}\Big(F_0^\theta(u)+G_0(\mathbb{E}\theta)\Big)$$

which ends the proof. □

5.2 The upper bound

Now we establish the upper bound in the definition of Γ-convergence:

Proposition 9. *For every u in $L^p(\mathcal{O},\mathbb{R}^3)$, there exists a sequence $(u_\varepsilon(\omega))_{\varepsilon<0}$ in $L^p(\mathcal{O},\mathbb{R}^3)$ such that for* $\mathbf{P}$*-almost every $\omega\in\Omega$, $u_\varepsilon(\omega)\rightharpoonup u$ and*

$$\limsup_{\varepsilon\to 0} H_\varepsilon(\omega,u_\varepsilon(\omega))\leq H(u). \tag{33}$$

Proof. One may assume $H(u)<+\infty$. For $\eta>0$ intended to go to zero, let v_η be a η-minimizer in the definition of $H(u)$:

$$H(u)\geq F_0^{v_\eta}(u)+G_0(v_\eta)-\eta.$$

It is easily seen that one may assume that $v_\eta\in\mathcal{C}(\overline{\mathcal{O}},\mathbb{R}^3)$. According to Proposition 3 there exists $u_{\eta,\varepsilon}(\omega)$ almost surely weakly converging to u in $L^p(\mathcal{O},\mathbb{R}^3)$ with $1_{T_\varepsilon\cap\mathcal{O}}u_{\eta,\varepsilon}(\omega)=v_\eta$ and such that for $\mathbf{P}$-almost every ω in Ω

$$\lim_{\varepsilon\to 0} F_\varepsilon^{v_\eta}(\omega,u_{\eta,\varepsilon}(\omega))=F_0^{v_\eta}(u). \tag{34}$$

On the other hand from (30) and (32) there exists $v_{\eta,\varepsilon}(\omega)$ in $W^{1,p}_{\Gamma_0}(\mathcal{O},\mathbb{R}^3)$ almost surely weakly converging to v_η in $L^p(\mathcal{O},\mathbb{R}^3)$ of the form

$$v_{\eta,\varepsilon}(\omega)=a(\omega,\frac{\hat{x}}{\varepsilon})\big[\frac{1}{\mathbb{E}}v_\eta(x)+\varepsilon\phi_1(\omega,\frac{\widehat{x}}{\varepsilon})\xi_1^\eta+\varepsilon\phi_2(\omega,\frac{\widehat{x}}{\varepsilon})\xi_2^\eta\big].$$

which satisfies for $\mathbf{P}$-almost every ω

$$\lim_{\varepsilon\to 0} G_\varepsilon(\omega,v_{\eta,\varepsilon}(\omega))=G_0(v_\eta). \tag{35}$$

From now on, we do not indicate the dependence of the functions $u_{\eta,\varepsilon}$ and $v_{\eta,\varepsilon}$ on ω. Combining (34) and (35) we infer

$$F_0^{v_\eta}(u)+G_0(v_\eta)=\lim_{\varepsilon\to 0}F_\varepsilon^{v_\eta}(\omega,u_{\eta,\varepsilon})+\lim_{\varepsilon\to 0}G_\varepsilon(\omega,v_{\eta,\varepsilon}). \tag{36}$$

let us set

$$\tilde{u}_{\eta,\varepsilon}=u_{\eta,\varepsilon}+\varepsilon\phi_1(\omega,\frac{\widehat{x}}{\varepsilon})\xi_1^\eta+\varepsilon\phi_2(\omega,\frac{\widehat{x}}{\varepsilon})\xi_2^\eta.$$

It is easily to check that

$$\lim_{\varepsilon\to 0}F_\varepsilon^{\tilde{u}_{\eta,\varepsilon}}(\omega,\tilde{u}_{\eta,\varepsilon})=\lim_{\varepsilon\to 0}F_\varepsilon^{v_\eta}(\omega,u_{\eta,\varepsilon})$$

so that (36) and the fact that $1_{T_\varepsilon\cap\mathcal{O}}\tilde{u}_{\eta,\varepsilon}=v_{\eta,\varepsilon}$ yields

$$\begin{aligned}F_0^{v_\eta}(u)+G_0(v_\eta) &= \lim_{\varepsilon\to 0}\Big(F_\varepsilon^{\tilde{u}_{\eta,\varepsilon}}(\omega,\tilde{u}_{\eta,\varepsilon})+G_\varepsilon(\omega,v_{\eta,\varepsilon})\Big)\\ &= \lim_{\varepsilon\to 0}H_\varepsilon(\omega,\tilde{u}_{\eta,\varepsilon})\end{aligned} \tag{37}$$

Clearly $\tilde{u}_{\eta,\varepsilon}\rightharpoonup u$ in $L^p(\mathcal{O},\mathbb{R}^3)$. We end the proof by letting $\eta\to 0$ and using a standard diagonalization argument. □

Collecting Propositions 8, 9 and according to the variational nature of the Γ-convergence we obtain

Theorem 6. *The problem*

$$(\mathcal{P}_{H_\varepsilon}) \qquad \inf\left\{H_\varepsilon(\omega,u)-\int_{\mathcal{O}} L.u\ dx : v\in L^p(\mathcal{O},\mathbb{R}^3)\right\}$$

almost surely converges to the problem

$$(\mathcal{P}_{H}) \qquad \min\left\{H(u)-\int_{\mathcal{O}} L.u\ dx : v\in L^p(\mathcal{O},\mathbb{R}^3)\right\}$$

in the sense of the Γ-convergence and, up to a subsequence, every sequence $(u_\varepsilon(\omega))_{\varepsilon>0}$ of ε-minimizers of $(\mathcal{P}_{H_\varepsilon})$ almost surely weakly converges in $L^p(\mathcal{O},\mathbb{R}^3)$ to a minimizer of $(\mathcal{P}_H)$.

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