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Non local effects induced by sources concentrated in a soft junction and the gradient concentration phenomenon

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NON LOCAL EFFECTS INDUCED BY SOURCES CONCENTRATED IN A SOFT JUNCTION AND THE GRADIENT CONCENTRATION PHENOMENON

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Abstract. We show that the variational limit of a soft ε-junction problem $(\mathcal{P}_\varepsilon)$ with sources concentrated in the junction, is non local. The non local part of the associated energy functional possesses an integral representation with respect to the Gradient Young-Concentration measures generated by sequences $(\bar{u}_\varepsilon)_{\varepsilon>0}$ of minimizers of $(\mathcal{P}_\varepsilon)$.

1. Introduction

This work is concerned with a soft thin junction problem whose internal energy functional is perturbed by a source $\mathcal{S}_\varepsilon$ concentrated in the layer $B_\varepsilon := \Sigma \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$, $\Sigma \subset \mathbb{R}^{N-1}$, i.e., the total energy of the physical system is of the form

$$F_\varepsilon(u) = \int_{\Omega \setminus B_\varepsilon} f(\nabla u)\ dx + \varepsilon \int_{B_\varepsilon} g(\nabla u)\ dx - \langle \mathcal{S}_\varepsilon, u \rangle$$

where $\Omega \subset \mathbb{R}^N$, $u : \Omega \longrightarrow \mathbb{R}$ runs through the space $W^{1,2}_{\Gamma_0}(\Omega)$ of Sobolev functions with null trace on a part Γ_0 of the boundary of Ω. The source $\mathcal{S}_\varepsilon$ suitably rescaled on the (rescaled) layer $B := \Sigma \times (-\frac{1}{2}, \frac{1}{2})$, is assumed to strongly converge to some $\mathcal{S}$ in the dual of the space $V(B) := \left\{ u \in L^2(\Omega) : \frac{\partial u}{\partial x_N} \in L^2(\Omega) \right\}$. In Section 4 of the paper we give a general example of such sources which are measures in B_ε. Sources of the form $c \frac{1}{L(\varepsilon)} \mathbb{1}_{B_\varepsilon}$ where c is any constant and $L(\varepsilon) \sim \varepsilon$, is a trivial example of measures satisfying this condition with $\mathcal{S} = \mathbb{1}_B$.

The Euler-Lagrange equation associated with the minimization problem

$$(\mathcal{P}_\varepsilon) \qquad \min_{u \in W^{1,2}_{\Gamma_0}} F_\varepsilon(u)$$

is given by the Dirichlet problem

$$\begin{cases} -\mathrm{div} \nabla_\xi \sigma_\varepsilon(x, \nabla u) = \mathcal{S}_\varepsilon & \text{on } \Omega \\ u = 0 & \text{on } \Gamma_0, \\ \frac{\partial u}{\partial x_N} = 0 & \text{on } \partial\Omega \setminus \Gamma_0 \end{cases}$$

where $\sigma_\varepsilon(x, \xi) := \mathbb{1}_{\Omega \setminus B_\varepsilon} f(\xi) + \varepsilon \mathbb{1}_{B_\varepsilon} g(\xi)$. Among the physical motivations of $(\mathcal{P}_\varepsilon)$ one may mention various applications to heat conduction or electrostatic problems subjected to concentrated sources in a layer B_ε and whose conductivity in B_ε is of order the small size of B_ε. One may also think of membrane problems with an exterior loading concentrated in B_ε occupied by a material with stiffness of order the small size of B_ε.

From the mathematical point of view, it is worth noticing that the source (or the loading) $\mathcal{S}_\varepsilon$ is a non L^2-continuous perturbation of the energy functional $\int_{\Omega\setminus B_\varepsilon} f(\nabla u)\,dx + \varepsilon\int_{B_\varepsilon} g(\nabla u)\,dx$. When the size ε of the layer goes to zero, fields u_ε of bounded energy develop a discontinuity through Σ and we show that the problem gives rise to a non local effect at the limit. The latter is the main novelty when regarding the various studies devoted to the asymptotic modelings of junction problems (see [2, 8, 6, 9] and references therein). More precisely, at the variational limit, the internal energy functional of the junction $\varepsilon\int_{B_\varepsilon} g(\nabla u)\,dx$ and the external energy of the source $\langle \mathcal{S}_\varepsilon, u\rangle$ are combined into a functional of the type $\inf_{\theta\in X(u)} H(\theta)$ where $H(\theta) := \int_B g^{\infty,2}(\hat{0}, \frac{\partial\theta}{\partial x_N})\,dx - \langle \mathcal{S}, \theta\rangle$, $g^{\infty,2}$ is the 2-recession function of g and θ runs through a suitable subspace $X(u)$ of $V(B)$ depending on the traces $u^\pm$ on Σ.

Such a junction problem with a source concentrated in the junction was considered in [3] in a one dimensional case in order to highlight and illustrate a gradient concentration phenomenon, but we were not able to express the variational limit problem. This paper illustrates the same phenomenon with a complete description of the limit problem in the sense of Γ-convergence (Theorem 3.3). We show that the sequence of minimizers of $(\mathcal{P}_\varepsilon)$ converges to a minimizer $\bar{u}$ of the limit problem and generates a gradient Young-concentration measure $\bar{\mu}$ that we analyse in the spirit of [3]. Moreover the non local part $\inf_{\theta\in X(\bar{u})} H(\theta)$ of the total energy possesses an integral representation with respect to the Young-concentration measure $\bar{\mu}$ i.e., in some sense the measure $\bar{\mu}$ allows us to localize the non local part of the total energy in $\Sigma\times\{\pm 1\}$(Theorem 5.5). This integral representation provides new bounds on the measure $\bar{\mu}$ (Corollary 5.6).

The paper is organized as follows: in Section 2 we fix notation and provide a detailed description of the problem $(\mathcal{P}_\varepsilon)$. Section 3 is devoted to the asymptotic analysis of $(\mathcal{P}_\varepsilon)$ in the sense of the Γ-convergence of the functional F_ε extended to $L^2(\Omega)$ equiped with its strong topology. In Section 4 we describe a large class of sources $\mathcal{S}_\varepsilon$ satisfying our suitable convergence condition. Finally Section 5 is concerned with the analysis of the gradient concentration phenomenon generated by sequences of minimizers of $(\mathcal{P}_\varepsilon)$.

2. Description of the minimization problem

Let $\varepsilon > 0$ be a small parameter. The reference configuration is a cylinder $\Omega := \Sigma\times(-r,r)$ (with $r>\varepsilon$), where Σ is a bounded domain in $\mathbb{R}^{N-1}$, $N\geq 2$, with Lipschitz boundary. For $x\in\mathbb{R}^N$ we sometimes write $x=(\hat{x},x_N)$ where $\hat{x}\in\mathbb{R}^{N-1}$. In all the paper, C denotes a non negative constant which does not depend on ε and may vary from line to line. We do not relabel the various considered subsequences and the symbols $\to$ and $\rightharpoonup$ denote various strong convergences and weak convergences respectively. We define the following sets:

- $B_\varepsilon := \Sigma\times(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})$;
- $B := \Sigma\times(-\frac{1}{2},\frac{1}{2})$;
- $\Omega_\varepsilon = \Omega\setminus\bar{B}_\varepsilon$;
- Γ_0 is a subset of the boundary $\partial\Omega$ of Ω such that $\mathrm{dist}(\overline{\Gamma}_0, \overline{\partial B_\varepsilon\cap\partial\Omega}) > 0$;
- we write Ω_ε^-, Ω_ε^+, Ω^-, Ω^+, B_ε^+ and B_ε^- for the sets $\Omega_\varepsilon\cap[x_N<0]$ and $\Omega_\varepsilon\cap[x_N>0]$, $\Omega\cap[x_N<0]$, $\Omega\cap[x_N>0]$ and $B_\varepsilon\cap[x_N>0]$, $B_\varepsilon\cap[x_N<0]$ respectively.

We will be concerned with the the following spaces:

. $W^{1,2}_{\Gamma_0}(\Omega_\varepsilon) := \{u \in W^{1,2}(\Omega_\varepsilon) : u = 0 \text{ on } \Gamma_0\}$;
. $W^{1,2}_{\Gamma_0}(\Omega) := \{u \in W^{1,2}(\Omega) : u = 0 \text{ on } \Gamma_0\}$;
. $W^{1,2}_{\Gamma_0}(\Omega\backslash\Sigma) := \{u \in W^{1,2}(\Omega \setminus \Sigma) : u = 0 \text{ on } \Gamma_0\}$, and for every $z \in W^{1,2}_{\Gamma_0}(\Omega\backslash\Sigma)$, $z^\pm$ will stand for the traces of z considered as a Sobolev function on Ω^+ and Ω^- respectively.

We say that a function $h : \mathbb{R}^N \longrightarrow \mathbb{R} \cup \{+\infty\}$ satisfies a growth condition of order 2 if there exist α and β in $\mathbb{R}^+$ such that

$$\alpha\,|\xi|^2 \le h(\xi) \le \beta(1 + |\xi|^2) \text{ for all } \xi \in \mathbb{R}^N.$$

We consider two convex functions $f, g : \mathbb{R}^N \longrightarrow \mathbb{R}$ satisfying a growth condition of order 2, and we assume that there exists a positively 2-homogeneous function $g^{\infty,2}$ satisfying

$$\big|g(\xi) - g^{\infty,2}(\xi)\big| \le \beta(1 + |\xi|^{2-\delta}) \text{ for all } \xi \in \mathbb{R}^N, \tag{2.1}$$

for some δ, $0 < \delta < 2$. Note that $g^{\infty,2}$ is the positively 2-homogeneous recession function of g, i.e.,

$$g^{\infty,2}(\xi) = \lim_{t\to+\infty} \frac{g(t\xi)}{t^2},$$

is convex and satisfies the same growth condition of order 2. We define the space

$$V(B_\varepsilon) := \left\{u \in L^2(B_\varepsilon) : \frac{\partial u}{\partial x_N} \in L^2(B_\varepsilon)\right\}$$

equipped with the norm

$$\|u\|_{V(B_\varepsilon)} := \Big(\int_{B_\varepsilon} |u|^2\,dx + \int_{B_\varepsilon} \Big|\frac{\partial u}{\partial x_N}\Big|^2\,dx\Big)^{\frac{1}{2}}$$

and we denote the duality bracket between the topological dual space $V'(B_\varepsilon)$ and $V(B_\varepsilon)$ by $\langle\ ,\ \rangle$. The considered total energy functional $F_\varepsilon : L^2(\Omega) \longrightarrow \mathbb{R} \cup \{+\infty\}$ is defined by

$$F_\varepsilon(u) = \begin{cases} \displaystyle\int_{\Omega_\varepsilon} f(\nabla u)\,dx + \varepsilon \int_{B_\varepsilon} g(\nabla u)\,dx - \langle \mathcal{S}_\varepsilon, u\rangle \text{ if } u \in W^{1,2}_{\Gamma_0}(\Omega) \\ +\infty \text{ otherwise,} \end{cases}$$

where $\mathcal{S}_\varepsilon$ is given in $V'(B_\varepsilon)$. Our aim is to describe the asymptotic behavior of the minimization problem

$$(\mathcal{P}_\varepsilon) \qquad \min\left\{F_\varepsilon : u \in L^2(\Omega)\right\},$$

namely, the limit of $\min\{F_\varepsilon : u \in L^2(\Omega)\}$ together with the limit of the minimizer $\bar{u}_\varepsilon$, and to identify the limit problem in the framework of Γ-convergence.

Let us consider the space $V(B) := \left\{u \in L^2(B) : \frac{\partial u}{\partial x_N} \in L^2(B)\right\}$ equipped with the norm

$$\|u\|_{V(B)} := \Big(\int_B |u|^2\,dx + \int_B \Big|\frac{\partial u}{\partial x_N}\Big|^2\,dx\Big)^{\frac{1}{2}}.$$

The linear continuous operator

$$\tau_\varepsilon : V(B) \longrightarrow V(B_\varepsilon)$$

is defined for every $x = (\hat{x}, x_N) \in B_\varepsilon$ by $\tau_\varepsilon(u)(\hat{x}, x_N) := u(\hat{x}, \frac{x_N}{\varepsilon})$ and the transposed operator

$$^T\tau_\varepsilon : V'(B_\varepsilon) \longrightarrow V'(B)$$

is defined for every $u \in V(B)$ by $\langle \mathcal{S}_\varepsilon, \tau_\varepsilon(u)\rangle = \langle ^T\mathcal{S}_\varepsilon, u\rangle$ (for shorten notation, $\langle ., .\rangle$ denotes as well the duality bracket between $V'(B_\varepsilon)$ and $V(B_\varepsilon)$ as the duality bracket between $V'(B)$ and $V(B)$).

We make the following assumption on the source $\mathcal{S}_\varepsilon$: there exists $\mathcal{S}$ in $V'(B)$ such that

$$^T\tau_\varepsilon \mathcal{S}_\varepsilon \text{ strongly converges to } \mathcal{S} \text{ in } V'(B).$$

Then, in order to identify the Γ-limit of the functional F_ε, it will be more convenient to write the functional F_ε as

$$F_\varepsilon(u) = \begin{cases} \displaystyle\int_{\Omega_\varepsilon} f(\nabla u)\, dx + \varepsilon^2 \int_B g(\hat{\nabla}\tau_\varepsilon^{-1} u, \frac{1}{\varepsilon}\frac{\partial \tau_\varepsilon^{-1} u}{\partial x_N})\, dx - \langle ^T\tau_\varepsilon \mathcal{S}_\varepsilon, \tau_\varepsilon^{-1} u\rangle \text{ if } u \in W^{1,2}_{\Gamma_0}(\Omega) \\ +\infty \text{ otherwise.} \end{cases}$$

3. The variational asymptotic model

Let $H : V(B) \longrightarrow \mathbb{R}$ be the functional defined by

$$H(\theta) := \int_B g^{\infty,2}(\hat{0}, \frac{\partial \theta}{\partial x_N})\, dx - \langle \mathcal{S}, \theta\rangle.$$

We are going to show that when $L^2(\Omega)$ is equiped with its strong topology, the functional F_ε Γ-converges to the functional $F_0 : L^2(\Omega) \longrightarrow \mathbb{R} \cup \{+\infty\}$ given by

$$F_0(u) = \begin{cases} \displaystyle\int_\Omega f(\nabla u)\, dx + \inf_{\theta \in X(u)} H(\theta) \text{ if } u \in W^{1,2}_{\Gamma_0}(\Omega \setminus \Sigma), \\ +\infty \text{ otherwise,} \end{cases}$$

where $X(u) := \big\{\theta \in V(B) : \theta(., \pm\frac{1}{2}) = u^\pm\big\}$.

Before addressing the variational convergence process, we begin by establishing some compactness properties for sequences with bounded energy. Let us introduce the ε-translate operator T_ε from $W^{1,2}(\Omega)$ into $W^{1,2}(\Omega \setminus \Sigma)$. For any function $w \in W^{1,2}(\Omega)$, $\tilde{w}$ stands for its extension by reflexion on $\Sigma \times (-2r, -r) \cup (r, 2r)$ and we define the ε-translate $T_\varepsilon w$ of w by

$$T_\varepsilon w(\hat{x}, x_N) = \begin{cases} \widetilde{w}(\hat{x}, x_N + \frac{\varepsilon}{2}) & \text{if } x \in \Omega^+; \\ \widetilde{w}(\hat{x}, x_N - \frac{\varepsilon}{2}) & \text{if } x \in \Omega^-. \end{cases}$$

Lemma 3.1 (compactness)**.** *Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence in $L^2(\Omega)$ such that $\sup_{\varepsilon>0} F_\varepsilon(u_\varepsilon) < +\infty$. Then*

(i)

$$\int_{B_\varepsilon} |u_\varepsilon|^2\, d\hat{x} \le C\varepsilon\Big(\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2\, dx + \varepsilon \int_{B_\varepsilon} |\frac{\partial u_\varepsilon}{\partial x_N}|^2\, dx\Big); \tag{3.1}$$

(ii)

$$\sup_{\varepsilon>0}\Big(\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx + \varepsilon \int_{B_\varepsilon} \left|\frac{\partial u_\varepsilon}{\partial x_N}\right|^2 dx\Big) < +\infty; \tag{3.2}$$

(iii) *there exist $u \in W^{1,2}_{\Gamma_0}(\Omega \setminus \Sigma)$ and a subsequence of $(u_\varepsilon)_{\varepsilon>0}$ such that $u_\varepsilon \to u$ in $L^2(\Omega)$ and $u_\varepsilon \rightharpoonup u$ in $W^{1,2}_{\Gamma_0}(\Omega_\eta)$ for all $\eta > 0$;*

(iv) *there exist $\theta \in V(B)$ and a subsequence such that $\tau_\varepsilon^{-1} u_\varepsilon \rightharpoonup \theta$ in $V(B)$, i.e.*

$$\tau_\varepsilon^{-1} u_\varepsilon \rightharpoonup \theta \text{ in } L^2(B),$$
$$\frac{\partial \tau_\varepsilon^{-1} u_\varepsilon}{\partial x_N} \rightharpoonup \frac{\partial \theta}{\partial x_N} \text{ in } L^2(B);$$

moreover, $\varepsilon \hat{\nabla} \tau_\varepsilon^{-1} u_\varepsilon \rightharpoonup 0$ in $L^2(B, \mathbb{R}^2)$;

(v) $\theta(., \pm\frac{1}{2}) = u^\pm$.

Proof. Proof of (i). Writting

$$u_\varepsilon(\hat{x}, x_N) = T_\varepsilon u_\varepsilon(\hat{x}, 0) + \int_{\frac{\varepsilon}{2}}^{x_N} \frac{\partial}{\partial x_N} u_\varepsilon(\hat{x}, t)\, dt,$$

an easy computation gives

$$\begin{aligned}
\int_{B_\varepsilon^+} |u_\varepsilon|^2 \, dx &\leq C\varepsilon \Big(\int_{\Omega^+} |\nabla T_\varepsilon u_\varepsilon|^2 + \varepsilon \int_{B\varepsilon^+} \left| \frac{\partial u_\varepsilon}{\partial x_N} \right|^2 \, dx \Big) \\
&\leq C\varepsilon \Big(\int_{\Omega_\varepsilon^+} |\nabla u_\varepsilon|^2 + \varepsilon \int_{B\varepsilon^+} \left| \frac{\partial u_\varepsilon}{\partial x_N} \right|^2 \, dx \Big).
\end{aligned}$$

The same holds in B_ε^- so that we get (3.1).

Proof of (ii). From the coercivity conditions satisfied by f and g, estimate (3.1), and the strong convergence of ${}^T\tau_\varepsilon \mathcal{S}_\varepsilon$ in $V(B)$, one has

$$\begin{aligned}
\alpha \Big(\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \, dx + \varepsilon \int_{B_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_N} \right|^2 dx \Big) &\leq C + |\langle \mathcal{S}_\varepsilon, u_\varepsilon \rangle| \\
&= C + |\langle {}^T\tau_\varepsilon \mathcal{S}_\varepsilon, \tau_\varepsilon^{-1} u_\varepsilon \rangle| \\
&\leq C + \left\| {}^T\tau_\varepsilon \mathcal{S}_\varepsilon \right\|_{V'(B)} \left\| \tau_\varepsilon^{-1} u_\varepsilon \right\|_{V(B)} \\
&= C + \left\| {}^T\tau_\varepsilon \mathcal{S}_\varepsilon \right\|_{V'(B)} \Big(\frac{1}{\varepsilon} \int_{B_\varepsilon} |u_\varepsilon|^2 \, dx + \varepsilon \int_{B_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_N} \right|^2 dx \Big)^{\frac{1}{2}} \\
&\leq C + C \Big(\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \, dx + \varepsilon \int_{B_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_N} \right|^2 dx \Big)^{1/2}.
\end{aligned}$$

Then, setting $X_\varepsilon := \Big(\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \, dx + \varepsilon \int_{B_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_N} \right|^2 dx \Big)^{1/2}$, (3.2) follows from the estimate $\alpha X_\varepsilon^2 \leq C + C X_\varepsilon$.

Proof of (iii).

Step 1. We claim that there exist $z \in W^{1,2}(\Omega \backslash \Sigma)$ and a subsequence of $(u_\varepsilon)_{\varepsilon > 0}$ such that $T_\varepsilon u_\varepsilon \rightharpoonup z$ in $W^{1,2}(\Omega \backslash \Sigma)$ and strongly in $L^2(\Omega \backslash \Sigma)$. Clearly,

$$T_\varepsilon u_\varepsilon \in W^{1,2}(\Omega \backslash \Sigma) \text{ and } \nabla T_\varepsilon u_\varepsilon = T_\varepsilon \nabla u_\varepsilon \text{ for all } \varepsilon > 0. \tag{3.3}$$

Combining the Poincaré inequality, (3.2) and (3.3), we deduce

$$\sup_{\varepsilon > 0} \|T_\varepsilon u_\varepsilon\|^2_{W^{1,2}(\Omega \backslash \Sigma, \mathbb{R}^N)} \leq C \sup_{\varepsilon > 0} \left(\int_{\Omega_\varepsilon} |\nabla u_\varepsilon(x)|^2 \, dx + \varepsilon \int_{B_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_N}(x) \right|^2 dx \right) < +\infty.$$

Therefore, $(T_\varepsilon u_\varepsilon)_{\varepsilon > 0}$ is bounded in $W^{1,2}(\Omega \backslash \Sigma)$ and the claim follows immediately.

Step 2. We establish that there exists u in $L^2(\Omega)$ such that we can extract of the previous subsequence $(u_\varepsilon)_{\varepsilon>0}$ a subsequence strongly converging to u in $L^2(\Omega)$. More precisely, we prove that $\lim\limits_{\varepsilon\to 0}\int_\Omega |u_\varepsilon(x)|^2\,dx = \int_\Omega |z(x)|^2\,dx$. We can write

$$\int_{\Omega_\varepsilon} |u_\varepsilon(x)|^2\,dx = \int_{\Omega^+\cup\Omega^-} |T_\varepsilon u_\varepsilon(x)|^2\,dx - \int_{\Sigma\times\left((r-\frac{\varepsilon}{2},r)\cup(-r,-r+\frac{\varepsilon}{2})\right)} |u_\varepsilon(x)|^2\,dx,$$

so that

(3.4)

$$\|u_\varepsilon\|^2_{L^2(\Omega)} = \int_{\Omega^+\cup\Omega^-} |T_\varepsilon u_\varepsilon(x)|^2\,dx + \int_{B_\varepsilon} |u_\varepsilon(x)|^2\,dx - \int_{\Sigma\times\left((r-\frac{\varepsilon}{2},r)\cup(-r,-r+\frac{\varepsilon}{2})\right)} |u_\varepsilon(x)|^2\,dx.$$

From (3.1), $\lim\limits_{\varepsilon\to 0}\int_{B_\varepsilon} |u_\varepsilon(x)|^2\,dx = 0$. On the other hand since $T_\varepsilon u_\varepsilon \to z$ in $L^2(\Omega)$, we infer

$$\lim_{\varepsilon\to 0} \|u_\varepsilon\|^2_{L^2(\Omega)} = \int_\Omega |z(x)|^2\,dx.$$

It remains to establish that

$$\lim_{\varepsilon\to 0} \int_{\Sigma\times\left((r-\frac{\varepsilon}{2},r)\cup(-r,-r+\frac{\varepsilon}{2})\right)} |u_\varepsilon(x)|^2\,dx = 0,$$

which is an easy consequence of the strong convergence of $T_\varepsilon u_\varepsilon$ to z in $L^2(\Omega)$.

Step 3. We show that $u = z$. Since $u_\varepsilon \rightharpoonup u$ in $L^2(\Omega)$ and $T_\varepsilon u_\varepsilon \rightharpoonup z$ in $W^{1,2}(\Omega\setminus\Sigma)$, we have for any $\varphi \in C_c^\infty(\Omega)$,

$$\begin{aligned}
\int_\Omega u(x)\varphi(x)dx &= \lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon(x)\varphi(\widehat{x}, x_N - \frac{\varepsilon}{2})dx \\
&= \lim_{\varepsilon\to 0}\int_\Omega T_\varepsilon u_\varepsilon(x)\varphi(x)dx \\
&= \int_\Omega z(x)\varphi(x)dx.
\end{aligned}$$

Thus, $u = z$, and we deduce that $u_\varepsilon \to u$ in $L^2(\Omega)$ and that $u \in W^{1,2}(\Omega\setminus\Sigma)$.

Step 4. It remains to establish that for any $\eta > 0$, there exists a subsequence of $(u_\varepsilon)_{\varepsilon>0}$ such that $u_{\varepsilon\lfloor\Omega_\eta} \rightharpoonup u_{\lfloor\Omega_\eta}$ in $W^{1,2}_{\Gamma_0}(\Omega_\eta)$. It will immediately result that $u \in W^{1,2}_{\Gamma_0}(\Omega\setminus\Sigma)$.

Let $\eta > 0$. Clearly, there exists $0 < \varepsilon_0 < \eta$ such that $\Omega_\eta \subseteq \Omega_\varepsilon$ for all $\varepsilon \le \varepsilon_0$. By the Poincaré inequality we have

$$\sup_{\varepsilon>0} \|u_\varepsilon\|^2_{W^{1,2}(\Omega_\eta,\mathbb{R}^3)} \le C \sup_{\varepsilon>0}\left(\int_{\Omega_\varepsilon} |\nabla u_\varepsilon(x)|^2\,dx + \varepsilon\int_{B_\varepsilon}\left|\frac{\partial u_\varepsilon}{\partial x_N}(x)\right|^2 dx\right) < +\infty.$$

Thus, $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $W^{1,2}_{\Gamma_0}(\Omega_\eta)$, and there exist $w \in W^{1,2}_{\Gamma_0}(\Omega_\eta)$ and a not relabelled subsequence of $(u_\varepsilon)_{\varepsilon>0}$ satisfying $u_\varepsilon \to w$ in $L^2(\Omega_\eta)$ and $u_\varepsilon \rightharpoonup w$ in $W^{1,2}_{\Gamma_0}(\Omega_\eta)$. It is easily seen that in fact $w = u_{\lfloor\Omega_\eta}$.

Proof of (iv). The weak convergence of $\tau_\varepsilon^{-1}u_\varepsilon$ to some θ in $V(B)$ follows from (3.1) and (3.2). Indeed

$$\begin{aligned}\sup_{\varepsilon>0}\left\|\tau_\varepsilon^{-1}u_\varepsilon\right\|_{V(B)} &= \sup_{\varepsilon>0}\Big(\frac{1}{\varepsilon}\int_{B_\varepsilon}|u_\varepsilon|^2\,dx+\varepsilon\int_{B_\varepsilon}\left|\frac{\partial u_\varepsilon}{\partial x_N}\right|^2 dx\Big)^{\frac12}\\ &\le C\sup_{\varepsilon>0}X_\varepsilon<+\infty.\end{aligned}$$

Now we deduce that $\hat\nabla\tau_\varepsilon^{-1}u_\varepsilon \rightharpoonup \hat\nabla\theta$ in the distributional sense so that $\varepsilon\hat\nabla\tau_\varepsilon^{-1}u_\varepsilon \rightharpoonup 0$ in the distributional sense. On the other hand, from the coercivity of g, $\varepsilon\hat\nabla\tau_\varepsilon^{-1}u_\varepsilon$ weakly converges to some $L^2(B,\mathbb{R}^2)$ function. Hence, $\varepsilon\hat\nabla u_\varepsilon \rightharpoonup 0$ in $L^2(B,\mathbb{R}^2)$.

Proof of (v). Note that $\theta(.,\pm\frac12)$ is well defined. Indeed, one has

$$V(B)\subset W^{1,2}\big((-\frac12,\frac12),L^2(\Sigma)\big)\subset\mathcal{C}\big([-\frac12,\frac12],L^2(\Sigma)\big).$$

Clearly, $\tau_\varepsilon^{-1}u_\varepsilon(\hat x,\pm\frac12)=(T_\varepsilon u_\varepsilon)^\pm(\hat x)$ (in the sense of traces on Σ of $W^{1,2}_{\Gamma_0}(\Omega\setminus\Sigma)$-functions) so that $\tau_\varepsilon^{-1}u_\varepsilon(\hat x,\pm\frac12)\to u^\pm$ in $L^2(\Sigma)$. On the other hand, since

$$\tau_\varepsilon^{-1}u_\varepsilon(\hat x,x_N)=\tau_\varepsilon^{-1}u_\varepsilon(\hat x,\pm\frac12)+\int_{\pm\frac12}^{x_N}\frac{\partial\tau_\varepsilon^{-1}u_\varepsilon}{\partial x_N}(\hat x,s)\,ds$$

for a.e. x in B, we infer that for all $\varphi\in\mathcal{C}_c(\Sigma)$,

$$\begin{aligned}(3.5)\quad \int_{-\frac12}^{\frac12}\int_\Sigma\tau_\varepsilon^{-1}u_\varepsilon(\hat x,x_N)\varphi(\hat x)\,dx &= \int_\Sigma(T_\varepsilon u_\varepsilon)^\pm(\hat x)\varphi(\hat x)d\hat x\\ &+\int_{-\frac12}^{\frac12}\int_\Sigma\int_{\pm\frac12}^{x_N}\frac{\partial\tau_\varepsilon^{-1}u_\varepsilon}{\partial x_N}(\hat x,s)\varphi(\hat x)dsdx.\end{aligned}$$

By passing to the limit in (3.5), we obtain

$$\int_{-\frac12}^{\frac12}\int_\Sigma\theta\varphi(\hat x)\,dx=\int_\Sigma u^\pm(\hat x)\varphi(\hat x)d\hat x+\int_{-\frac12}^{\frac12}\int_\Sigma\int_{\pm\frac12}^{x_N}\frac{\partial\theta}{\partial x_N}(\hat x,s)\varphi(\hat x)dsdx$$

from which we deduce

$$\int_\Sigma u^\pm(\hat x)\varphi(\hat x)d\hat x=\int_\Sigma\theta(\hat x,\pm\frac12)\varphi(\hat x)d\hat x.$$

Thus $\theta(.,\pm\frac12)=u^\pm$ almost everywhere in Σ. □

Lemma 3.2. *For every $u\in W^{1,2}_{\Gamma_0}(\Omega\backslash\Sigma)$, $\inf_{\theta\in X(u)}H(\theta)>-\infty$ and there exists $\theta(u)\in X(u)$ such that $\inf_{\theta\in X(u)}H(\theta)=H(\theta(u))$.*

Proof. The proof follows from standard arguments used in the direct method of the Calculus of Variation. □

As a consequence of Lemma 3.2, in its domain $W^{1,2}_{\Gamma_0}(\Omega\setminus\Sigma)$, the functional F_0 may be written

$$F_0(u)=\int_\Omega f(\nabla u)\,dx+H(\theta(u)).$$

Theorem 3.3 is the main result of this section.

Theorem 3.3. *The sequence $(F_\varepsilon)_{\varepsilon>0}$ Γ-converges to the functional F_0 when $L^2(\Omega)$ is equipped with its strong topology.*

The proof proceeds from the two following propositions.

Proposition 3.4. *For every $u \in L^2(\Omega)$ and every $(u_\varepsilon)_{\varepsilon>0}$ strongly converging to u in $L^2(\Omega)$ one has*

$$F_0(u) \leq \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon).$$

Proposition 3.5. *For every $u \in L^2(\Omega)$ there exists $(v_\varepsilon)_{\varepsilon>0}$ strongly converging to u in $L^2(\Omega)$ satisfying*

$$F_0(u) \geq \limsup_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon).$$

Proof of Proposition 3.4. Assume that $\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) < +\infty$. From Lemma 3.1 $u \in W^{1,2}_{\Gamma_0}(\Omega \setminus \Sigma)$ and there exists $\theta \in X(u)$ such that $\tau_\varepsilon^{-1} u_\varepsilon \rightharpoonup \theta$ in $V(B)$. Since ${}^T\tau_\varepsilon \mathcal{S}_\varepsilon \to \mathcal{S}$ in $V'(B)$, one has

$$\lim_{\varepsilon\to 0} \langle {}^T\tau_\varepsilon \mathcal{S}_\varepsilon, \tau_\varepsilon^{-1} u_\varepsilon \rangle = \langle \mathcal{S}, \theta \rangle. \tag{3.6}$$

On the other hand, since from Lemma 3.1, $u_\varepsilon \rightharpoonup u$ in $W^{1,2}_{\Gamma_0}(\Omega_\eta)$ for all $\eta > 0$, one has

$$\liminf_{\varepsilon\to 0} \int_{\Omega_\varepsilon} f(\nabla u_\varepsilon)\, dx \geq \int_\Omega f(\nabla u)\, dx. \tag{3.7}$$

Finally from (iv) of Lemma 3.1 and a standard lower semicontinuity argument

$$\begin{aligned}
& \liminf_{\varepsilon\to 0} \varepsilon^2 \int_B g(\hat\nabla \tau_\varepsilon^{-1} u_\varepsilon, \frac{1}{\varepsilon}\frac{\partial(\tau_\varepsilon^{-1}u_\varepsilon)}{\partial x_N})\, dx \\
\geq\ & \liminf_{\varepsilon\to 0} \left(\varepsilon^2 \int_B g(\hat\nabla \tau_\varepsilon^{-1} u_\varepsilon, \frac{1}{\varepsilon}\frac{\partial(\tau_\varepsilon^{-1}u_\varepsilon)}{\partial x_N})\, dx - \int_B g^{\infty,2}(\varepsilon\hat\nabla \tau_\varepsilon^{-1} u_\varepsilon, \frac{\partial(\tau_\varepsilon^{-1}u_\varepsilon)}{\partial x_N})\, dx \right) \\
& + \liminf_{\varepsilon\to 0} \int_B g^{\infty,2}(\varepsilon\hat\nabla \tau_\varepsilon^{-1} u_\varepsilon, \frac{\partial(\tau_\varepsilon^{-1}u_\varepsilon)}{\partial x_N})\, dx \\
\geq\ & \liminf_{\varepsilon\to 0} \left(\varepsilon^2 \int_B g(\hat\nabla \tau_\varepsilon^{-1} u_\varepsilon, \frac{1}{\varepsilon}\frac{\partial(\tau_\varepsilon^{-1}u_\varepsilon)}{\partial x_N})\, dx - \int_B g^{\infty,2}(\varepsilon\hat\nabla \tau_\varepsilon^{-1} u_\varepsilon, \frac{\partial(\tau_\varepsilon^{-1}u_\varepsilon)}{\partial x_N})\, dx \right) \\
& + \int_B g^{\infty,2}(0,0,\frac{\partial\theta}{\partial x_N})\, dx \\
\geq\ & \int_B g^{\infty,2}(0,0,\frac{\partial\theta}{\partial x_N})\, dx
\end{aligned} \tag{3.8}$$

provided that we establish

$$\lim_{\varepsilon\to 0} \left(\varepsilon^2 \int_B g(\hat\nabla \tau_\varepsilon^{-1} u_\varepsilon, \frac{1}{\varepsilon}\frac{\partial(\tau_\varepsilon^{-1}u_\varepsilon)}{\partial x_N})\, dx - \int_B g^{\infty,2}(\varepsilon\hat\nabla \tau_\varepsilon^{-1} u_\varepsilon, \frac{\partial\theta}{\partial x_N})\, dx \right) = 0. \tag{3.9}$$

Since $g^{\infty,2}$ is positively homogeneous of degree 2, and from (2.1), we have

$$\begin{aligned}
& \int_B \left| \varepsilon^2 g(\widehat\nabla \tau_\varepsilon^{-1} u_\varepsilon, \frac{1}{\varepsilon}\frac{\partial \tau_\varepsilon^{-1}u_\varepsilon}{\partial x_N}) - g^{\infty,2}(\varepsilon\widehat\nabla \tau_\varepsilon^{-1} u_\varepsilon, \frac{\partial(\tau_\varepsilon^{-1}u_\varepsilon)}{\partial x_N}) \right| dx \\
=\ & \varepsilon^2 \int_B \left| g(\widehat\nabla \tau_\varepsilon^{-1} u_\varepsilon, \frac{1}{\varepsilon}\frac{\partial \tau_\varepsilon^{-1}u_\varepsilon}{\partial x_N}) - g^{\infty,2}(\widehat\nabla \tau_\varepsilon^{-1} u_\varepsilon, \frac{1}{\varepsilon}\frac{\partial(\tau_\varepsilon^{-1}u_\varepsilon)}{\partial x_N}) \right| dx \\
\leq\ & C\varepsilon^2 \int_B \left[1 + \left| \widehat\nabla \tau_\varepsilon^{-1} u_\varepsilon \right|^{2-\delta} + \left| \frac{1}{\varepsilon}\frac{\partial(\tau_\varepsilon^{-1}u_\varepsilon)}{\partial x_N} \right|^{2-\delta} \right] dx.
\end{aligned}$$

Thus, by using Hölder's inequality (take $p = \frac{2}{2-\delta},\ q = \frac{2}{\delta}$) we deduce

$$\int_B \left| \varepsilon^2 g(\widehat{\nabla}\tau_\varepsilon^{-1}u_\varepsilon, \frac{1}{\varepsilon}\frac{\partial \tau_\varepsilon^{-1}u_\varepsilon}{\partial x_N}) - g^{\infty,2}(\varepsilon\widehat{\nabla}\tau_\varepsilon^{-1}u_\varepsilon, \frac{\partial(\tau_\varepsilon^{-1}u_\varepsilon)}{\partial x_N}) \right| dx \le C\varepsilon^\delta$$

which proves (3.9). The conclusion of Proposition 3.4 follows by collecting (3.6), (3.7) and (3.8). $\square$

Proof of Proposition 3.5. Let $u \in L^2(\Omega)$. We have to construct a sequence $(v_\varepsilon)_{\varepsilon>0}$ strongly converging to u in $L^2(\Omega)$ such that $\limsup_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon) \le F_0(u)$. If $F_0(u) = +\infty$, then $u \in L^2(\Omega)\backslash W^{1,2}_{\Gamma_0}(\Omega\backslash\Sigma)$, and clearly, for any sequence $(v_\varepsilon)_{\varepsilon>0}$ converging to u, $\limsup_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon) \le F_0(u)$ is true. Now, for the harder part, we assume $F_0(u) < +\infty$, then $u \in W^{1,2}_{\Gamma_0}(\Omega\backslash\Sigma)$ and

$$F(u) = \int_\Omega f(\nabla u(x))dx + \inf_{\theta\in X(u)} H(\theta).$$

To complete the proof, we construct from $\overline{\theta} := \theta(u)$, i.e., $H(\overline{\theta}) = \inf_{\theta\in X(u)} H(\theta)$, a sequence $(v_\varepsilon)_{\varepsilon>0}$ strongly converging to u in $L^2(\Omega)$ and satisfying

$$F_0(u) \ge \limsup_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon).$$

The proof is divided into four steps:

Step 1. Let us extend u and $\overline{\theta}$ by 0 in $(\mathbb{R}^{N-1}\setminus\Sigma)\times(-r,r)$ and write $\widetilde{u}$ and $\widetilde{\overline{\theta}}$ these extended functions. For each $\delta > 0$, set

$$u_\delta := \rho_\delta * \widetilde{u} \text{ defined by } \rho_\delta * \widetilde{u}(\widehat{x}, x_N) = \int_{\mathbb{R}^{N-1}} \rho_\delta(\widehat{x}-\widehat{y})\widetilde{u}(\widehat{y}, x_N)d\widehat{y} \quad \text{for all } (\widehat{x}, x_N) \in \Omega;$$
$$\theta_\delta := \rho_\delta * \widetilde{\overline{\theta}} \text{ defined by } \rho_\delta * \widetilde{\overline{\theta}}(\widehat{x}, x_N) = \int_{\mathbb{R}^{N-1}} \rho_\delta(\widehat{x}-\widehat{y})\widetilde{\overline{\theta}}(\widehat{y}, x_N)d\widehat{y} \quad \text{for all } (\widehat{x}, x_N) \in \Omega.$$

Clearly,

- $\theta_\delta(\widehat{x}, \pm\frac{1}{2}) = u_\delta(\widehat{x}, 0)$ for all $\widehat{x} \in \Sigma$
- $u_\delta \to u$ in $L^2(\Omega)$ and $\theta_\delta \to \overline{\theta}$ in $L^2(B)$
- $u_\delta \in W^{1,2}(\Omega)$ and $\theta_\delta \in W^{1,2}(B)$

Next, for each $\delta > 0$, we define the sequence $(v_{\delta,\varepsilon})_{\varepsilon>0}$ as follows:

$$v_{\delta,\varepsilon}(\widehat{x}, x_N) = \begin{cases} u_\delta(\widehat{x}, x_N \pm \dfrac{\varepsilon}{2}) & \text{on} \quad \Omega_\varepsilon^\mp \\ \\ \theta_\delta(\widehat{x}, \dfrac{x_N}{\varepsilon}) & \text{on} \quad B_\varepsilon. \end{cases} \tag{3.10}$$

Obviously $v_{\delta,\varepsilon}(\widehat{x}, x_N)$ belongs to $W^{1,2}(\Omega)$ and strongly converges to u_δ in $L^2(\Omega)$.

Step 2. We show that $\lim_{\varepsilon\to 0} F_\varepsilon(v_{\delta,\varepsilon}) = F_0(u_\delta) = \int_\Omega f(\nabla u_\delta)dx + H(\theta_\delta)$. In fact, we claim that

$$\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} f(\nabla v_{\delta,\varepsilon})(x)dx = \int_\Omega f(\nabla u_\delta)(x)dx \tag{3.11}$$

$$\lim_{\varepsilon\to 0} \left(\varepsilon^2 \int_B g(\widehat{\nabla}\tau_\varepsilon^{-1}v_{\delta,\varepsilon}, \frac{1}{\varepsilon}\frac{\partial\tau_\varepsilon^{-1}v_{\delta,\varepsilon}}{\partial x_N})(x)dx - \left\langle T_{\tau_\varepsilon}\mathcal{S}_\varepsilon, \tau_\varepsilon^{-1}v_{\delta,\varepsilon}\right\rangle \right) = H(\theta_\delta). \tag{3.12}$$

Proof of (3.11): one has

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} f(\nabla v_{\delta,\varepsilon})(x)dx &= \lim_{\varepsilon\to 0}\Big(\int_{\Omega_\varepsilon^+} f(\nabla u_\delta)(\widehat{x},x_N-\frac{\varepsilon}{2})dx+\int_{\Omega_\varepsilon^-} f(\nabla u_\delta(\widehat{x},x_N+\frac{\varepsilon}{2}))dx\Big)\\ &= \int_{\Omega^+} f(\nabla u_\delta)(x)dx+\int_{\Omega^-} f(\nabla u_\delta)(x)dx\\ &= \int_{\Omega} f(\nabla u_\delta)(x)dx.\end{aligned}$$

Proof of (3.12): Since $g^{\infty,2}$ is positively homogeneous of degree 2 and ${}^T\tau_\varepsilon\mathcal{S}_\varepsilon$ strongly converges to $\mathcal{S}$ in $V'(B)$, one has

$$\lim_{\varepsilon\to 0}\Big(\varepsilon^2\int_B g(\widehat{\nabla}\theta_\delta,\frac{1}{\varepsilon}\frac{\partial\theta_\delta}{\partial x_N})(x)dx-\langle {}^T\tau_\varepsilon\mathcal{S}_\varepsilon,\theta_\delta\rangle\Big)=\int_B g^{\infty,2}(\widehat{0},\frac{\partial\theta_\delta}{\partial x_N})dx-\langle\mathcal{S},\theta_\delta\rangle.$$

Step 3. We establish that $\lim_{\delta\to 0}F_0(u_\delta)=F_0(u)$. Since

$$\begin{aligned}F_0(u_\delta) &= \int_\Omega f(\nabla u_\delta)dx+H(\theta_\delta)\\ &= \int_\Omega f(\nabla u_\delta)dx+\int_B g^{\infty,2}(\widehat{0},\frac{\partial\theta_\delta}{\partial x_N})dx\end{aligned}$$

the result is a straightforward consequence of $u_\delta\to u$ in $L^2(\Omega)$ and $\theta_\delta\to\bar{\theta}$ in $L^2(B)$.

Step 4. By using a standard diagonalization argument, from step 2 and step 3, there exists a mapping $\varepsilon:\longrightarrow\delta(\varepsilon)$ such that $v_{\delta(\varepsilon)}\longrightarrow u$ in $L^2(\Omega)$ and $\lim_{\varepsilon\to 0}F_\varepsilon(v_{\delta(\varepsilon)})=F(u)$. The sequence $(v_\varepsilon)_{\varepsilon>0}$ where $v_\varepsilon:=v_{\delta(\varepsilon)}$ fullfils all the conditions except the boundary condition on Γ_0. By using De Giorgi's slicing method in a neighborhood of Γ_0, one can modify v_ε in Ω_ε into a function $\tilde{v}_\varepsilon$ satisfying the boundary condition, and from assumption $\mathrm{dist}(\overline{\Gamma}_0,\overline{\partial B_\varepsilon\cap\partial\Omega})>0$, which is equal to v_ε in B_ε, and satisfies $\liminf_{\varepsilon\to 0}\int_{\Omega_\varepsilon}f(\nabla v_\varepsilon)\,dx=\liminf_{\varepsilon\to 0}\int_{\Omega_\varepsilon}f(\nabla\tilde{v}_\varepsilon)\,dx$. Still denoting by v_ε this new function, we have $\lim_{\varepsilon\to 0}F_\varepsilon(v_\varepsilon)=F_0(u)$ and the proof is complete. □

Remark 3.6. In order to give an interpretation of the limit energy functional it is worthwhile to write

$$\inf_{\theta\in X(u)}H(\theta)=\inf_{\theta\in V_0(B)}\Big\{\int_B g^{\infty,2}(\hat{0},\frac{\partial\theta}{\partial x_N}(x)+[u](\hat{x}))\,dx-\langle\mathcal{S},\theta\rangle\Big\}-\langle\mathcal{S},\tilde{u}\rangle \tag{3.13}$$

where $[u]=u^+-u^-$, $V_0(B)=\big\{\theta\in V(B):\theta=0\text{ on }\Sigma\times\{\pm\frac{1}{2}\}\big\}$ and $\tilde{u}(x)=x_N[u](\hat{x})+\frac{u^+(\hat{x})+u^-(\hat{x})}{2}$. Thererfore when the limit source $\mathcal{S}$ vanishes on $V(B)$, by using Jensen's inequality, $\inf_{\theta\in X(u)}H(\theta)$ reduces to

$$\int_\Sigma g^{\infty,2}\big(\hat{0},[u](\hat{x})\big)\,d\hat{x}$$

which is nothing but the surface energy of the model obtained in [8]. In this case $G:u\mapsto\inf_{\theta\in X(u)}H(\theta)$ is a local functional with density h defined by $h(\hat{x})=g^{\infty,2}(\hat{0},[u](\hat{x}))$. By contrast when the limit source is not trivial, the functional G is non local in general and assumes the following form

$$G(u)=\int_B g^{\infty,2}(\hat{0},\frac{\partial\theta_{[u]}}{\partial x_N}(x)+[u](\hat{x}))\,dx-\langle\mathcal{S},\theta_{[u]}\rangle-\langle\mathcal{S},\tilde{u}\rangle$$

where $\theta_{[u]}$ is the minimizer of (3.13). In this general case, the functional G is a non local functional, not only of the jump field $[u]$, but also of the trace fields u^+ and u^-.

4. Examples of measure sources $\mathcal{S}_\varepsilon$ concentrated in B_ε

The general form of elements of $V'(B)$ is given for every θ in $V(B)$ by $\langle \mathcal{S}, \theta\rangle = \int_B s_0\theta\ dx + \int_B s_1 \frac{\partial\theta}{\partial x_N}\ dx$ where $(s_0, s_1) \in L^2(B) \times L^2(B)$. The limit sources $\mathcal{S}$ considered in this section are generated by measures $\mathcal{S}_\varepsilon$ in $\mathbb{M}(B_\varepsilon)$ whose slicing structure $\mathcal{H}^{N-1}\lfloor\Sigma \otimes \mathcal{S}^\varepsilon_{\hat{x}}$ is such that their slicing components $\mathcal{S}^\varepsilon_{\hat{x}}$ do not present a diffuse singular part in their Lebesgue-Nikodym decomposition in $\mathbb{M}(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$, i.e., are of the general form

$$\mathcal{S}^\varepsilon_{\hat{x}} = a_\varepsilon(\hat{x}, .)\ dt + \sum_{n=-\infty}^{+\infty} b_{\varepsilon,n}(\hat{x})\delta_{t^\varepsilon_n(\hat{x})}$$

where

$$\begin{cases} a_\varepsilon \in L^2(B_\varepsilon),\ b_{\varepsilon,n} \in L^2(\Sigma), \\ t^\varepsilon_n : \Sigma \longrightarrow (-\dfrac{\varepsilon}{2}, \dfrac{\varepsilon}{2}) \text{ is a Borel measurable map.} \end{cases}$$

Roughly, such sources $\mathcal{S}_\varepsilon$ are sum of a function a_ε in $L^2(B_\varepsilon)$ and a countable sum of surface sources, each of them being concentrated in the $N-1$-dimensional surface included in B_ε whose graph is t^ε_n. We make the following additional assumptions:

- (H1) there exists $a \in L^2(B)$ such that $\varepsilon\tau_\varepsilon^{-1}a_\varepsilon \to a$ in $L^2(B)$;
- (H2) there exists $b_n \in L^2(\Sigma)$ such that $b_{\varepsilon,n} \to b_n$ in $L^2(\Sigma)$ when $\varepsilon \to 0$;
- (H3) there exists $c_n \in \mathbb{R}^+$ such that $\|b_{\varepsilon,n}\|_{L^2(\Sigma)} \leq c_n$ and $\sum_{n=-\infty}^{+\infty} c_n < +\infty$;
- (H4) there exists $(t_n)_{n\in\mathbb{Z}}$ in $(-1/2, 1/2)$ such that $t^\varepsilon_n = \varepsilon t_n$ for all $n \in \mathbb{Z}$.

It is easy to check that the measure ${}^T\tau_\varepsilon\mathcal{S}_\varepsilon$ of $\mathbb{M}(B)$ is given by: ${}^T\tau_\varepsilon\mathcal{S}_\varepsilon = \mathcal{H}^{N-1}\lfloor\Sigma \otimes ({}^T\tau_\varepsilon\mathcal{S}^\varepsilon)_{\hat{x}}$ where

$$({}^T\tau_\varepsilon\mathcal{S}^\varepsilon)_{\hat{x}} = \varepsilon\tau_\varepsilon^{-1}a_\varepsilon\ dt + \sum_{n=-\infty}^{+\infty} b_{\varepsilon,n}(\hat{x})\delta_{t_n(\hat{x})}.$$

Proposition 4.1. *The measure ${}^T\tau_\varepsilon\mathcal{S}_\varepsilon$ strongly converges in $V'(B)$ to the measure $\mathcal{S}$ defined for every $\theta \in V(B)$ by*

$$\langle \mathcal{S}, \theta\rangle = \int_B a\theta\ dx + \sum_{n=-\infty}^{+\infty} \int_\Sigma b_n(\hat{x})\theta(\hat{x}, t_n(\hat{x}))\ d\hat{x}.$$

Therefore, the functional F_ε Γ-converges to the functional $F_0 : L^2(\Omega) \longrightarrow \mathbb{R}\cup\{+\infty\}$ given by

$$F_0(u) = \begin{cases} \displaystyle\int_\Omega f(\nabla u)\ dx + \inf_{\theta\in X(u)} \left\{ \int_B g^{\infty,2}(\hat{0}, \frac{\partial\theta}{\partial x_N})\ dx - \int_B a\theta\ dx - \sum_{n=-\infty}^{+\infty}\int_\Sigma b_n(\hat{x})\theta(\hat{x}, t_n(\hat{x}))\ d\hat{x} \right\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{if } u \in W^{1,2}_{\Gamma_0}(\Omega\setminus\Sigma) \\ +\infty \ \textit{otherwise.} \end{cases}$$

Proof. The second assertion is a straightforward consequence of Theorem 3.3 provided that we establish the strong convergence of ${}^T\tau_\varepsilon\mathcal{S}_\varepsilon$ to $\mathcal{S}$ in $V'(B)$. For every

$\theta \in V(B)$ we have

$$\langle {}^T\tau_\varepsilon \mathcal{S}_\varepsilon - \mathcal{S}, \theta \rangle = \int_B (\varepsilon \tau_\varepsilon^{-1} a_\varepsilon - a)\theta \; dx + \int_\Sigma \sum_{n=-\infty}^{+\infty} (b_{\varepsilon,n} - b_n)\theta(\hat{x}, t_n(\hat{x})) \; d\hat{x},$$

thus

(4.1)

$$\left| \langle {}^T\tau_\varepsilon \mathcal{S}_\varepsilon - \mathcal{S}, \theta \rangle \right| \leq \|\theta\|_{L^2(B)} \left\| \varepsilon \tau_\varepsilon^{-1} a_\varepsilon - a \right\|_{L^2(B)} + \sum_{n=-\infty}^{+\infty} \Big[\|b_{\varepsilon,n} - b_n\|_{L^2(\Sigma)} \Big(\int_\Sigma |\theta(\hat{x}, t_n(\hat{x})|^2 \, d\hat{x} \Big)^{\frac{1}{2}} \Big].$$

But it is easy to establish that there exists a non negative constant C such that

$$\Big(\int_\Sigma |\theta(\hat{x}, t_n(\hat{x})|^2 \, d\hat{x} \Big)^{\frac{1}{2}} \leq C \, \|\theta\|_{V(B)}$$

so that (4.1) yields

$$\left\| {}^T\tau_\varepsilon \mathcal{S}_\varepsilon - \mathcal{S} \right\|_{V'(B)} \leq \left\| \varepsilon \tau_\varepsilon^{-1} a_\varepsilon - a \right\|_{L^2(B)} + C \sum_{n=-\infty}^{+\infty} \|b_{\varepsilon,n} - b_n\|_{L^2(\Sigma)} \, .$$

The conclusion follows from assumptions $(H1)$, $(H2)$ and $(H3)$. □

5. The gradient concentration phenomenon

We first recall the notion of gradient Young-concentration measure introduced in [3]. Let us denote the unit sphere $\{-1, 1\}$ of $\mathbb{R}$ by $\mathbb{S}^0$, and consider $S \subset\subset \Sigma$, $B'_\varepsilon := S \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$.

Definition 5.1. *A pair $(v, \mu_S) \in L^2(\Omega) \times \mathbb{M}^+(\bar{\Omega} \times \mathbb{S}^0)$ is a gradient Young-concentration measure (localized on S) iff there exists a sequence $(v_\varepsilon)_{\varepsilon>0}$ in $W^{1,2}_{\Gamma_0}(\Omega)$ satisfying*

$$\begin{cases} \sup\limits_{\varepsilon>0} \displaystyle\int_{\Omega \setminus \Sigma_\varepsilon} |\nabla v_\varepsilon|^2 \; dx < +\infty, \\ v_\varepsilon \to v \text{ in } L^2(\Omega), \\ \mu_\varepsilon := \delta_{\frac{\partial v_\varepsilon}{\partial x_N} / \left| \frac{\partial v_\varepsilon}{\partial x_N} \right| \, (x)} \otimes \varepsilon \mathbb{1}_{B'_\varepsilon} \left| \frac{\partial v_\varepsilon}{\partial x_N} \right|^2 \; dx \stackrel{*}{\rightharpoonup} \mu_S. \end{cases}$$

We say that the sequence $(v_\varepsilon)_{\varepsilon>0}$ generates the gradient Young-concentration measure (v, μ). We denote the set of gradient Young-concentration measures localized on S by $\mathcal{YC}(S)$.

Recall that the weak convergence $\stackrel{*}{\rightharpoonup}$ above is defined by

$$\int_{B'_\varepsilon} \varepsilon \theta(x) \tilde{\varphi}(\frac{\partial v_\varepsilon}{\partial x_N}) \; dx \to \int_{\bar{\Omega}} \int_{\mathbb{S}^{m-1}} \theta(x) \varphi(\zeta) \; d\mu_S$$

for all $\theta \in \mathcal{C}(\bar{\Omega})$ and all $\varphi \in \mathcal{C}(\mathbb{S}^0)$, where the 2-homogeneous extension $\tilde{\varphi} : \mathbb{R} \to \mathbb{R}$ of $\varphi \in \mathcal{C}(\mathbb{S}^0)$ is defined for all $\zeta \in \mathbb{R}^m$ by

$$\tilde{\varphi}(\zeta) = \begin{cases} |\zeta|^2 \varphi(\frac{\zeta}{|\zeta|}), & \text{if } \zeta \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

In [3], Theorem 3.1, the gradient Young-concentration measures was characterized as follows.

Theorem 5.2 (Characterization)**.** *A pair $(v, \mu_S = \mu_x \otimes \pi)$ belongs to $\mathcal{YC}(S)$ if and only if $v \in W^{1,2}_{\Gamma_0}(\Omega \setminus \Sigma)$, π is concentrated on $\bar{S}$ and, for every $\varphi \in \mathcal{C}(\mathbb{S}^0)$ such that $\varphi^{**} > -\infty$,*

$$
(5.1) \qquad \begin{aligned} &\frac{d\pi}{d\mathcal{H}^{N-1}\lfloor S}(x) \int_{\mathbb{S}^0} \varphi(\zeta)\ d\mu_x \geq \varphi^{**}([v](x)) \quad \text{for } \mathcal{H}^{N-1} \text{ a. e. } x \in S \\ &\int_{\mathbb{S}^0} \varphi(\zeta)\ d\mu_x \geq 0 \quad \text{for } \pi_s \text{ a. e. } x \in \bar{S} \end{aligned}
$$

where $\pi = \frac{d\pi}{d\mathcal{H}^{N-1}\lfloor S}\mathcal{H}^{N-1}\lfloor S + \pi_s$ is the Radon-Nikodym decomposition of π with respect to the measure $\mathcal{H}^{N-1}\lfloor S$.

Remark 5.3. Although from (3.2), $\delta_{\frac{\partial v_\varepsilon}{\partial x_N}/\left|\frac{\partial v_\varepsilon}{\partial x_N}\right|(x)} \otimes \varepsilon \mathbb{1}_{B_\varepsilon} |\frac{\partial v_\varepsilon}{\partial x_N}|^2\ dx$ possesses weak cluster points in the sense of the weak convergence $\overset{*}{\rightharpoonup}$ made precise above, for technical reason (proof of the sufficient conditions in Proposition 3.5 in [3]), it was not possible to state such a characterization for these cluster points because of possible concentration effects on the boundary of Σ. This is the reason why we deal with gradient Young-concentration measures localized on $S \subset\subset \Sigma$.

Taking into account that the 2-homogeneous extension $\tilde{\varphi} : \mathbb{R} \to \mathbb{R}$ of $\varphi \in \mathcal{C}(\mathbb{S}^0)$ satisfying $\varphi^{**} > -\infty$ is of the form

$$
\varphi(\zeta) = \begin{cases} c\zeta^2 \text{ if } \zeta \geq 0 \\ d\zeta^2 \text{ if } \zeta \leq 0, \end{cases}
$$

with $(c, d) \in \mathbb{R}^+ \times \mathbb{R}^+$, the above characterization theorem can be reduced to the following (cf Corollary 3.6 in [3])

Corollary 5.4. *A measure $(v, \mu = (a(x)\delta_1 + b(x)\delta_{-1}) \otimes \pi)$ belongs to $\mathcal{YC}(S)$ if and only if $v \in W^{1,2}_{\Gamma_0}(\Omega \setminus \Sigma)$, π is concentrated on $\bar{S}$ and*

$$
\frac{d\pi}{d\mathcal{H}^{N-1}\lfloor S}(x)\big(a(x)c + b(x)d\big) \geq \varphi([v](x)) \text{ for } \mathcal{H}^{N-1}\lfloor S \text{ a.e. } x \text{ and for all } (c, d) \in \mathbb{R}^+ \times \mathbb{R}^+
$$

where $\varphi(\zeta) = \begin{cases} c\zeta^2 & \text{if } \zeta \geq 0 \\ d\zeta^2 & \text{if } \zeta \leq 0 \end{cases}$.

As stated in [3] Remark 2.5, every sequence $(u_\varepsilon)_{\varepsilon>0}$ satisfying (3.2) generates a gradient Young-concentration measure. Therefore every sequence $(\bar{u}_\varepsilon)_{\varepsilon>0}$, $\bar{u}_\varepsilon \in \operatorname{argmin} F_\varepsilon$, generates a measure $\bar{\mu}_S \in \mathcal{YC}(S)$. Let $\bar{u}$ be a strong limit of $(\bar{u}_\varepsilon)_{\varepsilon>0}$ in $L^2(\Omega)$, then, under the condition $g^{\infty,2}(\hat{\xi}, \xi_3) \geq g^{\infty,2}(\hat{0}, \xi_3)$, the next theorem states that the non local part $\inf_{\theta \in X(\bar{u})} H(\theta)$ of the total energy possesses an integral representation with respect to the Young-concentration measure $\bar{\mu}_S$. In some sense we localize the non local part on $S \times \{\pm 1\}$. Moreover, by using Theorem 5.2 we will deduce some bounds on $\bar{\mu}_S$.

Theorem 5.5. *Let $\bar{u}_\varepsilon$ be a minimizer of $\min\left\{F_\varepsilon(v) : v \in L^2(\Omega)\right\}$ and, for every $S \subset\subset \Sigma$, $(\bar{u}, \bar{\mu}_S)$ be a gradient Young-concentration measure localized on S generated by the sequence $(\bar{u}_\varepsilon)_{\varepsilon>0}$. Then the two following assertions hold:*

i) $\bar{u}_\varepsilon \to \bar{u}$ *in* $L^2(\Omega)$, $F_\varepsilon(\bar{u}_\varepsilon) \to F_0(\bar{u}) = \min\left\{F_0(u) : u \in L^2(\Omega)\right\}$;

ii) *Let $\mathcal{F}$ be a countable family of $S \subset\subset \Sigma$, then there exists $\bar\mu \in \mathbb{M}(\bar\Omega \times \mathbb{S}^0)$, $\bar\mu = \bar\mu_{\hat x} \otimes \bar\pi$ with $\bar\pi$ concentrated on $\bar\Sigma$ such that for all $S \in \mathcal{F}$, $\bar\mu \lfloor \bar S \times \mathbb{S}^0 = \bar\mu_S$. Assume furthermore that $g^{\infty,2}$ satisfies the condition*

$$\forall \xi_3 \in \mathbb{R}^3,\ g^{\infty,2}(\hat\xi, \xi_3) \geq g^{\infty,2}(\hat 0, \xi_3). \tag{5.2}$$

Then, every weak cluster point $\bar\theta$ of the sequence $(\tau_\varepsilon \bar u_\varepsilon)_{\varepsilon>0}$ in $V(B)$ satisfies $H(\bar\theta) = \inf_{\theta \in X(\bar u)} H(\theta)$ and

$$\begin{aligned} &\int_{-\frac12}^{\frac12} g^{\infty,2}\big(\hat 0, \frac{\partial\theta(\bar u)}{\partial x_N}\big)(\hat x, s)\ ds = \frac{d\bar\pi}{d\hat x}(\hat x) \int_{\mathbb{S}^0} g^{\infty,2}(\hat 0, \xi_3)\ d\bar\mu_{\hat x} \text{ for a.e. } \hat x \text{ in } S; \\ &\inf_{\theta \in X(\bar u)} H(\theta) = \int_\Sigma \Big[\frac{d\bar\pi}{d\hat x}(\hat x) \int_{\mathbb{S}^0} g^{\infty,2}(\hat 0, \xi_3)\ d\bar\mu_{\hat x}\Big]\ d\hat x - \langle \mathcal{S}, \bar\theta\rangle. \end{aligned} \tag{5.3}$$

Proof. According to the variational nature of the Γ-convergence, for a subsequence one has

$$\begin{aligned} &\bar u_\varepsilon \to \bar u \text{ in } L^2(\Omega) \\ &\lim_{\varepsilon\to0} F_\varepsilon(\bar u_\varepsilon) = F_0(\bar u) = \min\big\{F_0(v) : v \in L^2(\Omega)\big\} \\ &\qquad = \int_\Omega f(\nabla \bar u)\ dx + \inf_{\theta\in X(\bar u)} H(\theta). \end{aligned} \tag{5.4}$$

Fix $S \subset\subset \Sigma$. From (3.2), for the subsequence (possibly dependent on S) associated with the gradient Young-concentration measure $(\bar u, \bar\mu_S)$, there exist a subsequence and a measure $\bar\mu = \bar\mu_{\hat x} \otimes \bar\pi$ in $\mathbb{M}(\bar\Omega \times \mathbb{S}^0)$ with $\bar\pi$ concentrated in $\bar\Sigma$, such that

$$\delta_{\frac{\partial \bar u_\varepsilon}{\partial x_N}/\left|\frac{\partial \bar u_\varepsilon}{\partial x_N}\right|(x)} \otimes \varepsilon 1\!\!1_{B_\varepsilon} \left|\frac{\partial \bar u_\varepsilon}{\partial x_N}\right|^2 dx \rightharpoonup \bar\mu.$$

Thus, from (3.9) and (5.2) we infer

$$\begin{aligned} \lim_{\varepsilon\to0} \varepsilon^2 \int_B g\big(\widehat\nabla \tau_\varepsilon^{-1}\bar u_\varepsilon, \frac{1}{\varepsilon}\frac{\partial \tau_\varepsilon^{-1}\bar u_\varepsilon}{\partial x_N}\big)\ dx &= \lim_{\varepsilon\to0} \int_B g^{\infty,2}\big(\varepsilon\widehat\nabla\tau_\varepsilon^{-1}\bar u_\varepsilon, \frac{\partial\tau_\varepsilon^{-1}\bar u_\varepsilon}{\partial x_N}\big)\ dx \\ &\geq \lim_{\varepsilon\to0} \varepsilon \int_{B_\varepsilon} g^{\infty,2}\big(\hat 0, \frac{\partial \bar u_\varepsilon}{\partial x_N}\big)\ dx \\ &= \int_{\bar\Sigma} \Big(\int_{\mathbb{S}^0} g^{\infty,2}(\hat 0, \xi_3)\ d\bar\mu_{\hat x}\Big)\ d\bar\pi. \end{aligned} \tag{5.5}$$

Let $\bar\theta$ be the weak limit of $\tau_\varepsilon^{-1}\bar u_\varepsilon$ in $V(B)$ for the considered subsequence. Then, from (5.5), and since

$$\liminf_{\varepsilon\to0} \int_{\Omega_\varepsilon} f(\nabla \bar u_\varepsilon)\ dx \geq \int_\Omega f(\nabla\bar u)\ dx \text{ and } \lim_{\varepsilon\to0} \langle {}^T\tau_\varepsilon \mathcal{S}_\varepsilon, \tau_\varepsilon^{-1}\bar u_\varepsilon\rangle = \langle \mathcal{S}, \bar\theta\rangle,$$

we infer

$$\lim_{\varepsilon\to0} F_\varepsilon(\bar u_\varepsilon) \geq \int_\Omega f(\nabla\bar u)\ dx + \int_{\bar\Sigma}\Big(\int_{\mathbb{S}^0} g^{\infty,2}(\hat 0, \xi_3)\ d\bar\mu_{\hat x}\Big)\ d\bar\pi - \langle\mathcal{S}, \bar\theta\rangle. \tag{5.6}$$

Collecting (5.4) and (5.6) we obtain

$$\int_\Omega f(\nabla\bar u)\ dx + \inf_{\theta\in X(\bar u)} H(\theta) \geq \int_\Omega f(\nabla\bar u)\ dx + \int_{\bar\Sigma}\Big(\int_{\mathbb{S}^0} g^{\infty,2}(\hat 0, \xi_3)\ d\bar\mu_{\hat x}\Big)\ d\bar\pi - \langle\mathcal{S}, \bar\theta\rangle,$$

in particular

$$\int_\Omega f(\nabla\bar u)\ dx + H(\bar\theta) \geq \int_\Omega f(\nabla\bar u)\ dx + \int_{\bar\Sigma}\Big(\int_{\mathbb{S}^0} g^{\infty,2}(\hat 0, \xi_3)\ d\bar\mu_{\hat x}\Big)\ d\bar\pi - \langle\mathcal{S}, \bar\theta\rangle,$$

thus

$$\int_B g^{\infty,2}\big(\hat{0},\frac{\partial\bar\theta}{\partial x_N}\big)\ dx \ \geq\ \int_{\bar\Sigma}\Big(\int_{\mathbb{S}^0} g^{\infty,2}(\hat 0,\xi_3)\ d\bar\mu_{\hat x}\Big)\ d\bar\pi$$

$$\geq\ \int_{\bar\Sigma}\frac{d\bar\pi}{d\hat x}(\hat x)\Big(\int_{\mathbb{S}^0} g^{\infty,2}(\hat 0,\xi_3)\ d\mu_{\hat x}\Big)\ d\hat x. \tag{5.7}$$

On the other hand, by a standard lower semicontinuity argument, for every $\varphi \in \mathcal{C}_c(\Sigma)$, $\varphi \geq 0$,

$$\liminf_{\varepsilon\to 0}\int_B \varphi(\hat x) g^{\infty,2}\big(\hat 0,\frac{\partial\tau_\varepsilon^{-1}\bar u_\varepsilon}{\partial x_N}\big)\ dx \ =\ \int_\Sigma \varphi(\hat x)\Big(\int_{\mathbb{S}^0} g^{\infty,2}(\hat 0,\xi_3)\ d\bar\mu_{\hat x}\Big)\ d\bar\pi$$

$$\geq\ \int_B \varphi(\hat x) g^{\infty,2}\big(\hat 0,\frac{\partial\bar\theta}{\partial x_N}\big)\ dx$$

so that

$$\int_{-\frac12}^{\frac12} g^{\infty,2}\big(\hat 0,\frac{\partial\bar\theta}{\partial x_N}\big)(\hat x,s)\ ds \leq \frac{d\bar\pi}{d\hat x}(\hat x)\int_{\mathbb{S}^0} g^{\infty,2}(\hat 0,\xi_3)\ d\bar\mu_{\hat x} \quad \text{for a.e. } \hat x\in\Sigma. \tag{5.8}$$

Combining (5.7) and (5.8) we deduce

$$\int_{-\frac12}^{\frac12} g^{\infty,2}\big(\hat 0,\frac{\partial\bar\theta}{\partial x_N}\big)(\hat x,s)\ ds = \frac{d\bar\pi}{d\hat x}(\hat x)\int_{\mathbb{S}^0} g^{\infty,2}(\hat 0,\xi_3)\ d\bar\mu_{\hat x} \quad \text{for a.e. } \hat x\in\Sigma.$$

Clearly, $\bar\mu\lfloor\bar S\times\mathbb{S}^0 = \bar\mu_S$. Now, by using a standard Cantor's diagonal process, the same equality holds for all S of the countable familly $\mathcal{F}$. It remains to show that $H(\bar\theta) = \inf_{\theta\in X(\bar u)} H(\theta)$. It suffices to notice that

$$\begin{aligned}
\lim_{\varepsilon\to 0} F_\varepsilon(\bar u_\varepsilon) &= \int_\Omega f(\nabla\bar u)\ dx + \inf_{\theta\in X(\bar u)} H(\theta)\\
&\geq \liminf_{\varepsilon\to 0}\int_{\Omega_\varepsilon} f(\nabla\bar u_\varepsilon)\ dx + \liminf_{\varepsilon\to 0}\Big(\int_B g\big((\widehat\nabla\tau_\varepsilon^{-1}\bar u_\varepsilon,\frac{\partial\tau_\varepsilon^{-1}\bar u_\varepsilon}{\partial x_N}\big)\ dx - \langle{}^T\tau_\varepsilon\mathcal{S}_\varepsilon,\tau_\varepsilon^{-1}\bar u_\varepsilon\rangle\Big)\\
&\geq \int_\Omega f(\nabla\bar u)\ dx + \int_B g^{\infty,2}\big(\hat 0,\frac{\partial\bar\theta}{\partial x_N}\big)\ dx - \langle\mathcal{S},\bar\theta\rangle\\
&= \int_\Omega f(\nabla\bar u)\ dx + H(\bar\theta)
\end{aligned}$$

which completes the proof. □

We define the following two constants associated with the function g:

$$c(g) := \min\Big(\frac{g^{\infty,2}(\widehat 0,-1)}{g^{\infty,2}(\widehat 0,1)},\frac{g^{\infty,2}(\widehat 0,1)}{g^{\infty,2}(\widehat 0,-1)}\Big),\ C(g) = \frac{1}{c(g)} = \max\Big(\frac{g^{\infty,2}(\widehat 0,-1)}{g^{\infty,2}(\widehat 0,1)},\frac{g^{\infty,2}(\widehat 0,1)}{g^{\infty,2}(\widehat 0,-1)}\Big)$$

Recall that

$$g^{\infty,2}(\widehat 0,\xi) = \begin{cases} g^{\infty,2}(\widehat 0,-1)\,|\xi|^2 & \text{if} \quad \xi\leq 0\\ g^{\infty,2}(\widehat 0,1)\,|\xi|^2 & \text{if} \quad \xi > 0\end{cases}.$$

Moreover, from the assumption on the function g, clearly,

$$g^{\infty,2}(\widehat 0,1) > 0 \text{ and } g^{\infty,2}(\widehat 0,-1) > 0.$$

We make precise the probability measure $\bar\mu_{\hat x}$ localized on $S\subset\subset\Sigma$ as follows:

$$\bar\mu_{\hat x} := p(\hat x)\delta_1 + q(\hat x)\delta_{-1} \quad\text{with}\quad p(\hat x)+q(\hat x) = 1 \text{ a.e. } \hat x\in S.$$

Corollary 5.6. *Under assumptions of Theorem 5.5, the three following estimates hold:*

(i) *for a.e.* $\hat{x}$ *in* S

$$c(g)\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\hat{x},s)\right|^2\,ds\leq\frac{d\overline{\pi}}{dx_N}(\hat{x})\leq C(g)\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\hat{x},s)\right|^2\,ds,\tag{5.9}$$

and $\displaystyle\frac{d\overline{\pi}}{dx_N}(\hat{x})=\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\hat{x},s)\right|^2\,ds$ *when* $g^{\infty,2}(\widehat{0},-1)=g^{\infty,2}(\widehat{0},1)$*;*

(ii) $\displaystyle\frac{c(g)\ \left|[\overline{u}](\hat{x})\right|^2}{\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\hat{x},s)\right|^2\,ds}\leq p(\hat{x})\leq 1$ *for a.e.* $\hat{x}$ *such that* $[\overline{u}](\hat{x})>0$*;*

(iii) $\displaystyle\frac{c(g)\ \left|[\overline{u}](\hat{x})\right|^2}{\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\hat{x},s)\right|^2\,ds}\leq q(\hat{x})\leq 1$ *for a.e.* $\hat{x}$ *such that* $[\overline{u}](\hat{x})<0$*.*

Proof. Since $\overline{\mu}_{\hat{x}}=p(\widehat{x})\delta_1+q(\widehat{x})\delta_{-1}$, we have $\int_{\mathbb{S}^0}g^{\infty,2}(\xi)d\overline{\mu}_{\widehat{x}}=p(\widehat{x})g^{\infty,2}(\widehat{0},1)+q(\widehat{x})g^{\infty,2}(\widehat{0},-1)$ with $p(\widehat{x})+q(\widehat{x})=1$ a.e. $\widehat{x}$ in S so that from (5.3), one has

$$\begin{aligned}\int_{-\frac{1}{2}}^{\frac{1}{2}}g^{\infty,2}(\widehat{0},\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s))ds &= \left(\int_{\mathbb{S}^0}g^{\infty,2}(\xi)d\overline{\mu}_{\widehat{x}}\right)\frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x})\\ &= \frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x})\left\{p(\widehat{x})g^{\infty,2}(\widehat{0},1)+q(\widehat{x})g^{\infty,2}(\widehat{0},-1)\right\}\ \text{a.e. } \widehat{x}\in S.\end{aligned}\tag{5.10}$$

We are going to establish

$$c(g)\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds\leq\frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x})\leq C(g)\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds.$$

From (5.10) we deduce that

$$\begin{aligned}\min\left\{g^{\infty,2}(\widehat{0},-1),\ g^{\infty,2}(\widehat{0},1)\right\}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds &\leq \int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}}g^{\infty,2}(\widehat{0},\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s))ds\\ &= \left(\int_{\mathbb{S}^0}g^{\infty,2}(\xi)d\overline{\mu}_x\right)\frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x})\\ &= \left\{p(\widehat{x})g^{\infty,2}(\widehat{0},1)+q(\widehat{x})g^{\infty,2}(\widehat{0},-1)\right\}\frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x})\\ &\leq \max\left\{g^{\infty,2}(\widehat{0},-1),\ g^{\infty,2}(\widehat{0},1)\right\}\frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x})\end{aligned}\tag{5.11}$$

and

$$
\begin{aligned}
\min\left\{g^{\infty,2}(\widehat{0},-1),\ g^{\infty,2}(\widehat{0},1)\right\}\frac{d\pi}{d\widehat{x}}(\widehat{x}) &= \min\left\{g^{\infty,2}(\widehat{0},-1),\ g^{\infty,2}(\widehat{0},1)\right\}\{p(\widehat{x})+q(\widehat{x})\}\frac{d\pi}{d\widehat{x}}(\widehat{x}) \\
&\le \left\{p(\widehat{x})g^{\infty,2}(\widehat{0},1)+q(\widehat{x})g^{\infty,2}(\widehat{0},-1)\right\}\frac{d\pi}{d\widehat{x}}(\widehat{x}) \\
&= \left(\int_{\mathbb{S}^0} g^{\infty,2}(\xi)d\overline{\mu}_x\right)\frac{d\pi}{d\widehat{x}}(\widehat{x}) \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} g^{\infty,2}(\widehat{0},\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s))ds \\
&\le \max\left\{g^{\infty,2}(\widehat{0},-1),\ g^{\infty,2}(\widehat{0},1)\right\}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds \qquad (5.12)
\end{aligned}
$$

Then, from (5.11) and (5.12) we have

$$
c(g)\int_{\frac{-1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds = \frac{\min\left\{g^{\infty,2}(\widehat{0},-1),\ g^{\infty,2}(\widehat{0},1)\right\}}{\max\left\{g^{\infty,2}(\widehat{0},-1),\ g^{\infty,2}(\widehat{0},1)\right\}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds \le \frac{d\pi}{d\widehat{x}}(\widehat{x})
$$

and

$$
\frac{d\pi}{d\widehat{x}}(\widehat{x}) \le \frac{\max\left\{g^{\infty,2}(\widehat{0},-1),\ g^{\infty,2}(\widehat{0},1)\right\}}{\min\left\{g^{\infty,2}(\widehat{0},-1),\ g^{\infty,2}(\widehat{0},1)\right\}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds = C(g)\int_{\frac{-1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds
$$

from which we deduce,

$$
c(g)\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds \le \frac{d\pi}{d\widehat{x}}(\widehat{x}) \le C(g)\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds.
$$

Let us prove (ii) and (iii). According to Theorem 5.2, for every $\varphi \in \mathcal{C}(\mathbb{S}^0)$ such that $\varphi^{**} > -\infty$,

$$
\frac{d\overline{\pi}}{d\mathcal{H}^{N-1}_{\lfloor S}}(x)\int_{\mathbb{S}^0}\varphi(\zeta)\ d\overline{\mu}_x \ge \varphi^{**}([v](x)) \quad \text{for } \mathcal{H}^{N-1}\text{a.e. } x \in S, \tag{5.13}
$$

where $\overline{\pi} = \frac{d\overline{\pi}}{d\mathcal{H}^{N-1}_{\lfloor S}}\mathcal{H}^{N-1}_{\lfloor S} + \overline{\pi}_s$ is the Radon-Nikodym decomposition of $\overline{\pi}$ with respect to the measure $\mathcal{H}^{N-1}_{\lfloor S}$. We assume that $[\bar{u}](\hat{x}) > 0$ and show that

$$
\frac{c(g)\ |[\bar{u}](\hat{x})|^2}{\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\hat{x},s)\right|^2 ds} \le p(\hat{x}) \le 1.
$$

Let

$$
\varphi(\xi) = \begin{cases} \varphi(1)\,|\xi|^2 & \text{if} \quad \xi \ge 0 \\ 0 & \text{if} \quad \xi < 0 \end{cases}. \tag{5.14}
$$

Clearly, $\varphi^{**}([\bar{u}](\widehat{x})) = \varphi([\bar{u}](\widehat{x})) = \varphi(1)\ \left|[u](\widehat{x})\right|^2$. From the inequality (5.13), it follows that

$$\begin{aligned}
\varphi(1)\ \left|[\bar{u}](\widehat{x})\right|^2 &= \varphi^{**}([\bar{u}](\widehat{x})) \\
&\leq \frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x})\left(\int_{\mathbb{S}^0}\varphi(\xi)d\overline{\mu}_x\right) \\
&\leq C(g)\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds.\int_{\mathbb{S}^0}\varphi(\xi)d\overline{\mu}_x \\
&= C(g)p(\widehat{x})\varphi(1)\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds.
\end{aligned}$$

Then, we obtain

$$\frac{\left|[\bar{u}](\hat{x})\right|^2}{C(g)\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\hat{x},s)\right|^2 \ ds} = \frac{c(g)\ \left|[\bar{u}](\hat{x})\right|^2}{\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\hat{x},s)\right|^2 \ ds} \leq p(\hat{x}) \leq 1.$$

The proof of (iii) is similar. □

6. References

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