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by

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Abstract

By using ergodic theory and a variational process, we study the macroscopic behavior of a thin body with micro hight stiffness fibers randomly distributed according to a stationary point process. The thickness of the body, the stiffness and the size of the cross sections of the fibers depend on a small parameter ε. The variational limit functional energy obtained when ε tends to 0 is deterministic and depends on two variables: one is the deformation of a two dimensional body and describes the behavior of the medium in the matrix, the other captures the limit behavior of deformations in the fibers when the thickness, the stiffness and the size section become increasingly thin.

AMS *subject classifications*: 49J45, 74K15, 74C05, 74R20.

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Contents

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1 Introduction

In [19] we proposed a deterministic model of a randomly reinforced material with reference configuration, an open cylinder $\mathcal{O} = \hat{\mathcal{O}} \times (0,h)$, $\hat{\mathcal{O}} \subset \mathbb{R}^2$ including randomly distributed thin rigid fibers $T_\varepsilon(\omega) = \varepsilon D(\omega)\times(0,h)$. Our objective was to provide a simplified but accurate model of the slices of the geomaterial TexSol^{TM} ([14, 16, 17]). Let us recall that this soil reinforcement process mixes the soil (sand) with a wire and that the obtained material has a better mechanical resistance than the sand without wire. In our simplified model we assumed the wire to cut the surface perpendicularly so that the thin parallel cylinders, randomly distributed, represent the pieces of the wire which are perfectly stuck with a hyperelastic matrix which represent the sand. This hypothesis is acceptable when the thickness h is small. The main objection of this model is that h is fixed so that we did not say how h is small by comparison with the size ε of the wires section. In this paper we go back to the asymptotic analysis studied in [19] when the thickness h of the slice goes to zero with ε. We propose a two dimensional deterministic model which is a first attempt in the scope of non linear elasticity at obtaining a variational equivalent model of a very thin slices of randomly reinforced materials like Texsol^{TM}. We do not assume the basic condition on the elastic densities which ensures the principle of material frame-indifference. Furthermore our model does not take into account the a.e. injectivity of the deformation maps, and the necessity of an infinite amount of energy to compress a finite volume into zero volume. We hope to address this issue in future papers.

The open cylinder $\mathcal{O}_{h(\varepsilon)} := \widehat{\mathcal{O}} \times (0,h(\varepsilon))$ of $\mathbb{R}^3$, whose basis is a domain $\widehat{\mathcal{O}}$ of $\mathbb{R}^2$, is a reference configuration for a random fibered structure which may be described as follows. For $\varepsilon > 0$ we consider the union of cylinders $T_\varepsilon(\omega) := \varepsilon D(\omega) \times \mathbb{R}$ where $D(\omega) := \bigcup_{i\in\mathbb{N}} D(\omega_i)$ and $D(\omega_i)$ are disks distributed at random in $\mathbb{R}^2$ following a stochastic point process $\omega = (\omega_i)_{i\in\mathbb{N}}$ of $\mathbb{R}^2$ associated with a suitable probability space $(\Omega, \mathcal{A}, \mathbf{P})$. The random fibered structure is then given by $\mathcal{O} = \big(\mathcal{O}_{h(\varepsilon)} \setminus T_\varepsilon(\omega)\big) \cup \big(\mathcal{O}_{h(\varepsilon)} \cap T_\varepsilon(\omega)\big)$ (Figure 1).

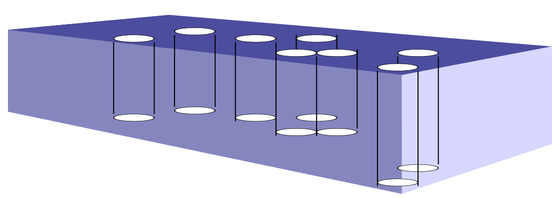

Figure 1: A slice of randomly fibered body of thikness $h(\varepsilon)$

We assume for instance the wire to be clamped on its lower sections and submitted to a surface loading on its upper sections. The material in the matrix is only submitted to a volume loading (the force of gravity). The equilibrium is described by the unique solution $\bar{u}_\varepsilon(\omega,.)$ of the variational problem $(\mathcal{P}_{E_{\varepsilon,h(\varepsilon)}})$

$$\inf_{u\in W^{1,p}_\varepsilon(\mathcal{O}_{h(\varepsilon)},\mathbb{R}^3)} \left\{ \int_{\mathcal{O}_{h(\varepsilon)}\setminus T_\varepsilon} f(\nabla u)\, dx + \frac{1}{\varepsilon^a}\int_{\mathcal{O}_{h(\varepsilon)}\cap T_\varepsilon} g(\nabla u)\, dx - \Phi_\varepsilon(u) \right\}$$

$$\Phi_\varepsilon(u) := \int_{\mathcal{O}_{h(\varepsilon)}\setminus T_\varepsilon} \mathcal{L}_\varepsilon.u\, dx - \int_{\widehat{\mathcal{O}}\cap\varepsilon D} \ell_\varepsilon(\hat{x},h(\varepsilon)).u(\hat{x},h(\varepsilon))\, d\hat{x}$$

where f, g are two quasiconvex functions,

$$W^{1,p}_\varepsilon(\mathcal{O}_{h(\varepsilon)},\mathbb{R}^3) := \left\{ u \in W^{1,p}(\mathcal{O}_{h(\varepsilon)},\mathbb{R}^3) : u = 0 \text{ on } \hat{\mathcal{O}} \cap \varepsilon D(\omega) \right\},$$

$\frac{1}{\varepsilon^a}$ stands for the high stiffness of the wire, and $\mathcal{L}_\varepsilon$, ℓ_ε denote the loadings in $L^{\frac{p}{p-1}}(\mathcal{O} \setminus T_\varepsilon, \mathbb{R}^3)$ and the topological dual of $W^{1,1-\frac{1}{p}}(\widehat{\mathcal{O}} \cap \varepsilon D, \mathbb{R}^3)$ respectively. For short we sometimes drop ω for the notation and, for instance, write T_ε and D instead of $T_\varepsilon(\omega)$ and $D(\omega)$.

Let denote by $\bar{u}_\varepsilon(\omega, .)$ a minimizer of $(\mathcal{P}_{E_{\varepsilon,h(\varepsilon)}})$. In this work we intend to study the behavior of $\overline{\overline{u_\varepsilon}}(\omega, .)$ defined by $\overline{\overline{u_\varepsilon}}(\omega, x) = \bar{u}_\varepsilon\big(\omega, \hat{x}, h(\varepsilon)x_3\big)$, which is minimizer of the problem $(\mathcal{P}_{E_\varepsilon})$

$$\inf_{u \in W^{1,p}_\varepsilon(\mathcal{O},\mathbb{R}^3)} \left\{ h(\varepsilon) \int_{\mathcal{O}\setminus T_\varepsilon} f(\hat{\nabla} u, \frac{1}{h(\varepsilon)} \frac{\partial u}{\partial x_3})\ dx + \frac{h(\varepsilon)}{\varepsilon^a} \int_{\mathcal{O}\cap T_\varepsilon} g(\hat{\nabla} u, \frac{1}{h(\varepsilon)} \frac{\partial u}{\partial x_3})\ dx - \Psi_\varepsilon(u) \right\}$$

$$\Psi_\varepsilon(u) = \int_{\mathcal{O}\setminus T_\varepsilon} L_\varepsilon.u\ dx - \int_{\widehat{\mathcal{O}}\cap \varepsilon D} l_\varepsilon.u(\hat{x}, 1))\ d\hat{x},$$

where $\mathcal{O} = \hat{\mathcal{O}} \times (0, 1)$, $W^{1,p}_\varepsilon(\mathcal{O}, \mathbb{R}^3) := \Big\{u \in W^{1,p}(\mathcal{O}, \mathbb{R}^3) : u = 0 \text{ on } \hat{\mathcal{O}} \cap \varepsilon D(\omega)\Big\}$ and L_ε, l_ε are the rescaled loading given by $L_\varepsilon(x) = h(\varepsilon)\mathcal{L}_\varepsilon\big(\hat{x}, h(\varepsilon)x_3\big)$ and $l_\varepsilon(\hat{x}, 1) = \ell_\varepsilon(\hat{x}, h(\varepsilon))$ respectively. The loading L_ε is assumed to be independent of ε and denoted by L while l_ε is assumed to be possibly very high, precisely of the form $\varepsilon^{-b}l$ for some l in $L^{\frac{p}{p-1}}(\widehat{\mathcal{O}}, \mathbb{R}^3)$. We perform the asymptotic analysis of $(\mathcal{P}_{E_\varepsilon})$ under the conditions $p > 1$, $a > 0$ and $h(\varepsilon) = \varepsilon^p$. For the surface loading we assume $b \leq p - 1 + \frac{a}{p}$ and we will see that the interesting case is $b = p - 1 + \frac{a}{p}$.

Let us denote by $\hat{Y}$ the unit cell of $\mathbb{R}^2$, by $f^{\infty,p}$ the p-recession function of the function f and, for all $\lambda \in \mathbf{M}^{3\times 2}$, set $\widehat{f^{\infty,p}}(\lambda) := \inf_{\xi \in \mathbb{R}^3} f^{\infty,p}\big((\lambda|\xi)\big)$ where we identify the set of 3×1-matrices with $\mathbb{R}^3$. For all $s \in \mathbb{R}^3$, let denote by $f_0(s)$ the almost sure limit when $n \to +\infty$ of

$$\inf_{w \in W^{1,p}_0(n\hat{Y}\setminus D(\omega),\mathbb{R}^3)} \left\{ \fint_{n\hat{Y}} \widehat{f^{\infty,p}}(\nabla w)\ d\hat{x} : \fint_{n\hat{Y}} w\ d\hat{x} = s \right\},$$

whose existence is ensured by an ergodic theorem in Section 2.3. Then we establish that $\overline{\overline{u_\varepsilon}}(\omega, .)$ almost surely weakly converges in $L^p(\mathcal{O}, \mathbb{R}^3)$ to $\overline{\overline{u}}$ satisfying for a.e. $\hat{x} \in \hat{\mathcal{O}}$ (Corollary 2.1)

$$\overline{\overline{u}}(\hat{x}) \in \partial f_0^*\Big(\int_0^1 L(\hat{x}, t)\ dt\Big).$$

In the case when $f = \frac{1}{2}|\ .\ |^2$ an easy computation yields (see Section 3.4)

$$\overline{\overline{u}}(\hat{x}) = \frac{\Lambda}{2} \int_0^1 L(\hat{x}, t)\ dt$$

where Λ is defined as follows: consider $U_n(\omega, .)$ solving the scalar random Dirichlet problem

$$\begin{cases} -\Delta U = 1 \text{ in } n\hat{Y} \setminus D(\omega), \\ U \in W^{1,2}_0(n\hat{Y} \setminus D(\omega)), \end{cases}$$

and set $\Lambda_n(\omega) := \int_{n\hat{Y}} U_n(\omega, .)\ d\hat{x}$. Then one can show that $\Lambda_n(\omega)$ almost surely converges when n tends to $+\infty$ toward a deterministic limit that we denote by Λ. This last result has to be compared with the Darcy's law in fluid mechanics (see [3, 10]) and may be considered as a Darcy's law in the solid mechanics framework.

Another significant fact is that $\mathbb{1}_{\mathcal{O}\cap T_\varepsilon}\overline{\overline{u_\varepsilon}}(\omega, .)$ and $\mathbb{1}_{\mathcal{O}\cap T_\varepsilon}\frac{\partial\overline{\overline{u_\varepsilon}}(\omega,.)}{\partial x_3}$ strongly converge to 0 in $L^p(\mathcal{O}, \mathbb{R}^3)$. We want to clarify these convergences: at what rate do these two functions converge to zero? To explore this question, denoting by $g^{\infty,p}$ the p-recession function of the function g, we establish that almost surely, $\varepsilon^{1-p-\frac{a}{p}}\mathbb{1}_{\mathcal{O}\cap T_\varepsilon}\overline{\overline{u_\varepsilon}}(\omega, .)$ and $\varepsilon^{1-p-\frac{a}{p}}\mathbb{1}_{\mathcal{O}\cap T_\varepsilon}\frac{\partial\overline{\overline{u_\varepsilon}}(\omega,.)}{\partial x_3}$ weakly converge in $L^p(\mathcal{O}, \mathbb{R}^3)$ toward $\bar{v}$ and $\frac{\partial \bar{v}}{\partial x_3}$

respectively, where $\bar{v}$ is the unique solution of the problem

$$\begin{cases} -\dfrac{\partial}{\partial x_3}\Big(D(g^{\infty,p})^{\perp}(\dfrac{\partial v}{\partial x_3})\Big) = 0 \text{ in } \mathcal{O}, \\ v(\hat{x},0) = 0 \text{ on } \widehat{\mathcal{O}}, \\ D(g^{\infty,p})^{\perp}(\frac{\partial v}{\partial x_3}).e_3 = \theta^{p-1}\tilde{l} \text{ on } \widehat{\mathcal{O}} + e_3, \end{cases}$$

where

$$\tilde{l} = \begin{cases} l & \text{if } b = p-1+\frac{a}{p} \\ 0 & \text{if } b < p-1+\frac{a}{p}, \end{cases}$$

$g^{\infty,p}$ is the p-recession of the function g, $(g^{\infty,p})^{\perp}(s) := g^{\infty,p}(0,s)$ for all $s \in \mathbb{R}$, and $\theta = \displaystyle\int_{\Omega} |\hat{Y} \cap D(\omega)|\, d\mathbf{P}(\omega)$ is the asymptotic volume fraction of the fibers (Corollary 2.1).

We perform these two asymptotic behaviors thanks to a variational convergence method (related to the Γ-convergence) of the sole total energy functional

$$E_\varepsilon(\omega,u) := h(\varepsilon)\int_{\mathcal{O}\setminus T_\varepsilon} f(\hat{\nabla}u, \frac{1}{h(\varepsilon)}\frac{\partial u}{\partial x_3})\, dx + \frac{h(\varepsilon)}{\varepsilon^a}\int_{\mathcal{O}\cap T_\varepsilon} g(\hat{\nabla}u, \frac{1}{h(\varepsilon)}\frac{\partial u}{\partial x_3})\, dx - \Psi_\varepsilon(u)$$

of the problem $(\mathcal{P}_{E_\varepsilon})$ (Theorem 2.2).

2 The problem statement

2.1 Probabilistic setting

For all $x = (x_1, x_2, x_3)$ of $\mathbb{R}^3$, $\hat{x}$ stands for (x_1, x_2) and we denote by $\hat{Y}$ the unit cell $(0,1)^2$ of $\mathbb{R}^2$. For any $\delta > 0$ and any set $\hat{A}$ of $\mathbb{R}^2$, we make use of the following notation: $\hat{A}_\delta := \Big\{x \in \hat{A} : d(x, \mathbb{R}^2 \setminus \hat{A}) > \delta\Big\}$. For any bounded Borel set A of $\mathbb{R}^2$ or $\mathbb{R}^3$, $|A|$ denotes its Lebesgue measure and $\#(A)$ its cardinal when it is finite.

Let d be a given number satisfying $0 < d \leq 1$ and consider the set

$$\Omega = \big\{(\omega_i)_{i\in\mathbb{N}} : \omega_i \in \mathbb{R}^2,\ |\omega_i - \omega_j| \geq d \text{ for } i \neq j\big\}$$

equipped with the trace σ-algebra $\mathcal{A}$ of the standard product σ-algebra on Ω. Let $\hat{B}_{d/2}(0)$ denote the open ball of $\mathbb{R}^2$ centered at 0 with radius $d/2$, then for every $\omega = (\omega_i)_{(i)\in\mathbb{N}}$ we form the disk $D(\omega_i) := \omega_i + \hat{B}_{d/2}(0)$ and consider $D(\omega) := \bigcup_{i\in\mathbb{N}} D(\omega_i)$. Therefore $\omega \mapsto T(\omega) = D(\omega) \times \mathbb{R}$ is a random set in $\mathbb{R}^3$, union of random cylinders, whose basis is the union of the pairwise disjoint disks $D(\omega_i)$ of $\mathbb{R}^2$ centered at ω_i. We set $T_\varepsilon(\omega) := \varepsilon D(\omega) \times \mathbb{R}$. For every $z \in \mathbf{Z}^2$ we define the operator $\tau_z : \Omega \to \Omega$ by $\tau_z\omega = \omega - z$. Note that $D(\tau_z\omega) = D(\omega) - z$.

We assume that there exists a probability measure on $(\Omega, \mathcal{A})$ which satisfies the system of three following axioms:

(A_1) *Non sparsely distribution*: $\mathbf{P}\Big(\Big\{\omega \in \Omega : |\hat{Y} \cap D(\omega)| > 0\Big\}\Big) = 1$;

(A_2) *Stationary condition*: $\forall z \in \mathbf{Z}^2$, $\tau_z\#\mathbf{P} = \mathbf{P}$ where $\tau_z\#\mathbf{P}$ denotes the probability image of $\mathbf{P}$ by τ_z;

(A_3) *Asymptotic mixing property*: for all sets E and F of $\mathcal{A}$, $\lim_{|z|\to+\infty} \mathbf{P}(\tau_z E \cap F) = \mathbf{P}(E)\mathbf{P}(F)$.

Remark 2.1. *i) It would be more natural to consider stationary condition (A_2) with respect to the continuous group $(\tau_t)_{t\in\mathbb{R}^2}$ defined in the same way by $\tau_t\omega = \omega - t$. Actually the discrete group $(\tau_z)_{z\in\mathbf{Z}^3}$ suffices for the mathematical analysis. The size of the cell $\hat{Y}$ is chosen in such a way to fix the generator of the group $(\tau_z)_{z\in\mathbf{Z}^2}$. Condition (A_2) then says that every random function X taking its source in Ω is statistically homogeneous in the sense that X and $X\circ\tau_z$ have the same law (i.e. $X\#\mathbf{P} = X\circ\tau_z\#\mathbf{P}$). Roughly speaking, moving a window $\hat{A}$ in $\mathbb{R}^2$ following the translations in $\mathbb{R}^2$, the distributions of cross sections in the window are statistically the same.*

ii) Condition (A_1) together with condition (A_2) yield that the random set $D(\omega)$ is statistically not too sparse in $\mathbb{R}^2$. Indeed for every $\mathbf{Z}^2$-translated $\hat{A} = \hat{Y} + z$ of $\hat{Y}$

$$\begin{aligned}\mathbf{P}\big(\Big\{\omega : |\hat{A}\cap D(\omega)| > 0\Big\}\big) &= \mathbf{P}\big(\Big\{\omega : |\hat{Y}\cap(D(\omega)-z)| > 0\Big\}\big)\\ &= \mathbf{P}\big(\Big\{\omega : |\hat{Y}\cap(D(\tau_z\omega))| > 0\Big\}\big)\\ &= \mathbf{P}\big(\Big\{\omega : |\hat{Y}\cap(D(\omega))| > 0\Big\}\big) = 1.\end{aligned}$$

iii) Condition (A_3) says that the events $\tau_z E$ and F are independent provided that z be large enough.

iv) Consider $\bar{\omega} = (\bar{\omega}_i)_{i\in\mathbb{N}}$ where $\bar{\omega}_i$ are the centers of the hexagonal close-packing of disks in $\mathbb{R}^2$. Then $\bar{\omega}$ is a "maximal" distribution in the sense that $|\hat{Y}\cap D(\omega)| \le |\hat{Y}\cap D(\bar{\omega})|$ for a.s. ω in Ω.

A simple specimen of probability space which fulfills all the conditions above is the generalized random chessboard described bellow.

Example 2.1 (Random chessboard-like)**.** Given $d > 0$ as previously, let us consider a countable set of points $\Omega_0 = \{x_k : k\in\mathbb{N}\}$ in $\hat{Y}_{d/2}$ and set $\Omega := \Pi_{z\in\mathbf{Z}^2}\Omega_z$ where $\Omega_z = \Omega_0 + z$ for all $z\in\mathbf{Z}^2$. We equip Ω with the σ-algebra $\mathcal{A}$ generated by the cylinders of Ω. For a given family $(\alpha_k)_{k\in\mathbb{N}}$ of non negative numbers satisfying $\sum_{k\in\mathbb{N}}\alpha_k = 1$ we consider the probability measure $\mu_0 = \sum_{k\in\mathbb{N}}\alpha_k\delta_{x_k}$ on Ω_0 and the product probability measure $\mathbf{P} = \Pi_{z\in\mathbf{Z}}\mu_z$ on $(\Omega,\mathcal{A})$ where $\mu_z = \mu_0$ for all $z\in\mathbf{Z}$. Then it is easy to check that $\mathbf{P}$ satisfies axioms (A_1)-(A_3).

Remark 2.2. *All the results of the paper remain valid if we substitute for the disk $\hat{B}_{d/2}(0)$, any connex compact set of $\mathbb{R}^2$ included in $\hat{B}_{d/2}(0)$ and chosen at random.*

We would stress that, keeping the same probabilistic framework, but substituting now $\hat{B}_{r_\varepsilon d/2}$, $r_\varepsilon\to 0$ for $\hat{B}_{d/2}$, for critical growth of $r(\varepsilon)$, the limit problem seems to leads to a model which takes into account the random capacity of $D(\omega)$. We aim to treat this case in a forthcoming paper.

2.2 Functional analysis setting

We are given two quasiconvex functions f and g defined on $\mathbb{R}^3$ satisfying the standard growth condition of order $p > 1$: there exist two positive constants α,β, such that $\forall\zeta$ in $\mathbf{M}^{3\times 3}$

$$\alpha|\zeta|^p \le f(\zeta) \le \beta(1+|\zeta|^p), \tag{2.1}$$

idem for g. Note that f satisfies automatically the Lipschitz property

$$|f(\zeta) - f(\zeta')| \le \ell|\zeta-\zeta'|(1+|\zeta|^{p-1}+|\zeta'|^{p-1}) \tag{2.2}$$

for all $(\zeta,\zeta')\in\mathbf{M}^{3\times 3}\times\mathbf{M}^{3\times 3}$ where ℓ is a positive constant, idem for g. Furthermore, we assume that there exist $\beta' > 0$, $0 < r < p$ and a p-positively homogeneous function $f^{\infty,p}$ (the p-recession function of f) such that for all $\zeta\in\mathbf{M}^{3\times 3}$

$$|f(\zeta) - f^{\infty,p}(\zeta')| \le \beta'(1+|\zeta|^{p-r}). \tag{2.3}$$

From (2.3) we infer $\lim_{t\to+\infty}\dfrac{f(t\zeta)}{t^p}=f^{\infty,p}(\zeta)$ so that from (2.1), $f^{\infty,p}$ satisfies for all $\zeta\in\mathbf{M}^{3\times3}$

$$\alpha|\zeta|^p\le f^{\infty,p}(\zeta)\le\beta|\zeta|^p \tag{2.4}$$

and

$$|f^{\infty,p}(\zeta)-f^{\infty,p}(\zeta')|\le\ell|\zeta-\zeta'|(|\zeta|^{p-1}+|\zeta'|^{p-1}) \tag{2.5}$$

for all $(\zeta,\zeta')\in\mathbf{M}^{3\times3}\times\mathbf{M}^{3\times3}$. Finally $\lambda\mapsto\widehat{f^{\infty,p}}(\lambda):=\inf_{\xi\in\mathbb{R}^3}f^{\infty,p}\big((\lambda|\xi)\big)$ is assumed to be a convex function in the set $\mathbf{M}^{3\times2}$ of 3×2-matrices.

We define the p-recession function $g^{\infty,p}$ of g as in (2.3) and, for all s in $\mathbb{R}^3$, the function $(g^{\infty,p})^\perp$ by

$$(g^{\infty,p})^\perp(s)=\inf_{\xi\in\mathbf{M}^{3\times2}}g^{\infty,p}(\xi|s),$$

and we assume that $\forall s\in\mathbb{R}^3,\ (g^{\infty,p})^\perp(s)=g^{\infty,p}(0,s)$. Note that since $g^{\infty,p}$ is a rank 1 convex function, the density $(g^{\infty,p})^\perp$ is a convex function.

We consider the problem

$$(\mathcal{P}_{H_{\varepsilon,h(\varepsilon)}})\qquad\inf\big\{H_{\varepsilon,h(\varepsilon)}(\omega,u)-\langle\mathcal{L}_\varepsilon,u\rangle : u\in L^p(\mathcal{O},\mathbb{R}^3)\big\}$$

where the functional energy H_ε is given in $L^p(\mathcal{O},\mathbb{R}^3)$, $p>1$ by

$$H_{\varepsilon,h(\varepsilon)}(\omega,u)=\begin{cases}\displaystyle\int_{\mathcal{O}_{h(\varepsilon)}\setminus T_\varepsilon}f(\nabla u)\,dx+\frac{1}{\varepsilon^a}\int_{\mathcal{O}_{h(\varepsilon)}\cap T_\varepsilon}g(\nabla u)\,dx & \text{if } u\in W^{1,p}_\varepsilon(\mathcal{O}_{h(\varepsilon)},\mathbb{R}^3)\\ +\infty \text{ otherwise}\end{cases}$$

and

$$W^{1,p}_\varepsilon(\mathcal{O}_{h(\varepsilon)},\mathbb{R}^3):=\Big\{u\in W^{1,p}(\mathcal{O}_{h(\varepsilon)},\mathbb{R}^3):u=0\text{ on }\hat{\mathcal{O}}\cap D_\varepsilon(\omega)\Big\}.$$

The loadings $\mathcal{L}_\varepsilon$ and ℓ_ε satisfiy the following behavior: there exist L in $L^q(\mathcal{O},\mathbb{R}^3)$, l in $L^q(\widehat{\mathcal{O}},\mathbb{R}^3)$, $q=\frac{p}{p-1}$, and b in $\mathbb{R}$ such that

$$\mathcal{L}_\varepsilon\approx\frac{1}{h(\varepsilon)}L_1\Big(\hat{x},\frac{x_3}{h(\varepsilon)}\Big)\text{ in }\mathcal{O}_{h(\varepsilon)}\setminus T_\varepsilon$$
$$\ell_\varepsilon\approx\varepsilon^{-b}l(\hat{x},1)\text{ on }\widehat{\mathcal{O}}\cap\varepsilon D.$$

Let us introduce the change of scale $x_3=h(\varepsilon)y_3$. We want to study the behavior of $\overline{\overline{u_\varepsilon}}(\omega,\hat{x},x_3):=\bar{u}_\varepsilon(\omega,\hat{x},h(\varepsilon)x_3)$ where $\tilde{u}_\varepsilon$ is a minimizer of $(\mathcal{P}_{H_{\varepsilon,h(\varepsilon)}})$. Clearly $\overline{\overline{u_\varepsilon}}(\omega,\hat{x},x_3)$ is a minimizer of

$$(\mathcal{P}_{H_\varepsilon})\qquad\inf\left\{H_\varepsilon(\omega,u)-\int_{\mathcal{O}\setminus T_\varepsilon}L.u\,dx-\varepsilon^{-b}\int_{\widehat{\mathcal{O}}\cap\varepsilon D}l(\hat{x},1).u(\hat{x},1)\,d\hat{x}:u\in L^p(\mathcal{O},\mathbb{R}^3)\right\}$$

where

$$H_\varepsilon(\omega,u)=\begin{cases}\displaystyle h(\varepsilon)\int_{\mathcal{O}\setminus T_\varepsilon}f(\hat{\nabla}u,\frac{1}{h(\varepsilon)}\frac{\partial u}{\partial x_3})\,dx+h(\varepsilon)\varepsilon^{-a}\int_{\mathcal{O}\cap T_\varepsilon}g(\hat{\nabla}u,\frac{1}{h(\varepsilon)}\frac{\partial u}{\partial x_3})\,dx & \text{if } u\in W^{1,p}_\varepsilon(\mathcal{O},\mathbb{R}^3)\\ +\infty\text{ otherwise,}\end{cases}$$

$W^{1,p}_\varepsilon(\mathcal{O},\mathbb{R}^3):=\Big\{u\in W^{1,p}(\mathcal{O},\mathbb{R}^3):u=0\text{ on }\hat{\mathcal{O}}\cap\varepsilon D(\omega)\Big\}$. We denote by $E_\varepsilon(\omega,.)$ the total random energy defined for all $u\in L^p(\mathcal{O},\mathbb{R}^3)$ by

$$E_\varepsilon(\omega,u)=H_\varepsilon(\omega,u)-\int_{\mathcal{O}\setminus T_\varepsilon}L.u\,dx-\varepsilon^{-b}\int_{\widehat{\mathcal{O}}\cap\varepsilon D}l(\hat{x},1).u(\hat{x},1)\,d\hat{x}.$$

For the asymptotic analysis, we assume the following conditions:

$$a > 0, \ h(\varepsilon) = \varepsilon^p.$$

The condition $a > 0$ is introduced in order to supply a model with hight stiffness in the random cylinders, the reason of the choice $h(\varepsilon) = \varepsilon^p$ will appear in the next section. Roughly speaking, it is the good choice that provides sufficient information on the behavior of any sequence $(u_\varepsilon)_{\varepsilon>0}$ of bounded energy, i.e., statisfying $\sup_{\varepsilon>0} E_\varepsilon(\omega, .) < +\infty$ (cf Lemma 2.1 in the next section). The rescalled internal energy becomes now

$$H_\varepsilon(\omega, u) = \begin{cases} \varepsilon^p \displaystyle\int_{\mathcal{O}\setminus T_\varepsilon} f(\hat{\nabla} u, \frac{1}{\varepsilon^p}\frac{\partial u}{\partial x_3})\ dx + \varepsilon^{p-a} \int_{\mathcal{O}\cap T_\varepsilon} g(\hat{\nabla} u, \frac{1}{\varepsilon^p}\frac{\partial u}{\partial x_3})\ dx \text{ if } u \in W^{1,p}_\varepsilon(\mathcal{O}, \mathbb{R}^3) \\ +\infty \text{ otherwise.} \end{cases}$$

We sometimes consider each two following functionals

$$F_\varepsilon(\omega, u) = \begin{cases} \varepsilon^p \displaystyle\int_{\mathcal{O}\setminus T_\varepsilon} f(\hat{\nabla} u, \frac{1}{\varepsilon^p}\frac{\partial u}{\partial x_3})\ dx \text{ if } u \in W^{1,p}_{\Gamma_0}(\mathcal{O}, \mathbb{R}^3) \\ +\infty \text{ otherwise,} \end{cases}$$

and

$$G_\varepsilon(\omega, u) = \begin{cases} \varepsilon^{p-a} \displaystyle\int_{\mathcal{O}\cap T_\varepsilon} g(\hat{\nabla} u, \frac{1}{\varepsilon^p}\frac{\partial u}{\partial x_3})\ dx \text{ if } u \in W^{1,p}_{\Gamma_0}(\mathcal{O}, \mathbb{R}^3) \\ +\infty \text{ otherwise,} \end{cases}$$

so that $H_\varepsilon(\omega, .) = F_\varepsilon(\omega, .) + G_\varepsilon(\omega, .)$ in $L^p(\mathcal{O}, \mathbb{R}^3)$.

Let us set $\gamma := p - 1 + \dfrac{a}{p}$ (note that $\gamma > 0$). We will distinguish the following two cases for which the limit loading changes:

(C_1) $b = \gamma$;

(C_2) $b < \gamma$.

2.3 The limit densities

We are going to define the limit density energy associated with the functional $F_\varepsilon(\omega, .)$ by defining a suitable subadditive process defined in the probabilistic space $(\Omega, \mathcal{A}, \mathbf{P})$ governed by axioms (A_1)-(A_3). Let $\mathcal{I}$ denotes the set of all open intervals (a, b) of the lattice spanned by $\hat{Y}$. For all $A \in \mathcal{I}$ and all s in $\mathbb{R}^3$ set

$$\begin{aligned} \mathcal{S}_{\hat{A}}(\omega, s) &:= \inf \left\{ \int_{\hat{A}\setminus\overline{D(\omega)}} \widehat{f^{\infty,p}}(\hat{\nabla} w(\hat{x}))\ d\hat{x} : w \in \mathrm{Adm}_{\hat{A}}(\omega, s) \right\}, \\ \mathrm{Adm}_{\hat{A}}(\omega, s) &:= \left\{ w \in W^{1,p}_0(\hat{A} \setminus \overline{D(\omega)}, \mathbb{R}^3) : \fint_{\hat{A}} w\ dx = s, w = 0 \ \text{ on } \hat{A} \cap D(\omega) \right\}. \end{aligned}$$

Note that the random set $D(\omega)$ is not necessarily included in $\hat{A}$. It is standard to see that the random functionals defined in the introduction are measurable when $\Omega \times L^p(\mathcal{O}, \mathbb{R}^3)$ is equipped with the product σ-algebra $\mathcal{A} \otimes \mathcal{B}$ where $\mathcal{B}$ is the Borel σ-algebra associated with the normed space $L^p(\mathcal{O}, \mathbb{R}^3)$. Consequently, for all fixed $\hat{A}$ in $\mathcal{I}$ and all fixed s in $\mathbb{R}^3$, the map $\omega \mapsto \mathcal{S}_{\hat{A}}(\omega, s)$ is measurable.

For each fixed s in $\mathbb{R}^3$, it is easily seen that $\mathcal{S}(., s)$ satisfies the subadditivity condition: for every $I \in \mathcal{I}$ such that there exists a finite family $(I_j)_{j\in J}$ of disjoint intervals in $\mathcal{I}$ with $|I \setminus \bigcup_{j\in J} I_j| = 0$,

$$\mathcal{S}_I(., s) \leq \sum_{j\in J} \mathcal{S}_{I_j}(., s).$$

Moreover $\mathcal{S}(., s)$ is clearly covariant with respect to the group $(\tau_z)_{z\in\mathbf{Z}^2}$, i.e., for all $\hat{A} \in \mathcal{I}$ and all $z \in Z^2$,

$$\mathcal{S}_{\hat{A}+z}(., s) = \mathcal{S}_{\hat{A}}(., s) \circ \tau_z.$$

Actually we have

Theorem 2.1. *For all fixed s in $\mathbb{R}^3$ the map*

$$\begin{aligned} \mathcal{S}(.,s): \qquad \mathcal{I} &\longrightarrow L^1(\Omega,\mathcal{A},\mathbf{P}) \\ \hat{A} &\longmapsto \mathcal{S}_{\hat{A}}(.,s) \end{aligned}$$

is a subadditive process with respect to the group $(\tau_z)_{z\in\mathbf{Z}^2}$ defined by $\tau_z(\omega)=\omega-z$. It satisfies for all $s\in\mathbb{R}$, all $\hat{A}\in\mathcal{I}$ and all $\delta>0$ small enough

$$\mathcal{S}_{\hat{A}}(\omega,s)\le C(p)\frac{|s|^p}{\delta^p\left|(\hat{Y}\setminus D(\bar{\omega}))_{2\delta}\right|}|\hat{A}| \tag{2.6}$$

where $C(p)$ is a non negative constant depending only of p. Therefore for any regular family $(I_n)_{n\in\mathbb{N}}$ of sets in $\mathcal{I}$ the limit $\lim_{n\to\infty}\dfrac{\mathcal{S}_{I_n}(\omega,s)}{|I_n|}$ exists for $\mathbf{P}$ almost every $\omega\in\Omega$ and

$$\begin{aligned} \lim_{n\to\infty}\frac{\mathcal{S}_{I_n}(\omega,s)}{|I_n|} &= \inf_{m\in\mathbb{N}^*}\left\{\mathbf{E}\frac{\mathcal{S}_{[0,m[^2}(.,s)}{m^2}\right\} \\ &= \lim_{n\to+\infty}\left\{\mathbf{E}\frac{\mathcal{S}_{[0,n[^2}(.,s)}{n^2}\right\}. \end{aligned}$$

We denote by $f_0(s)$ the common value above.

Proof. Reproduce the proof of Theorem 2.2 in [19] with minor change of notation. □

Proposition 2.1. *The function f_0 is a positively homogeneous convex function of degree p, satisfies the growth conditions (2.4) with the same constant α, with a constant β possibly different, and satisfies the Lipschitz condition (2.5) with a constant L possibly different.*

Proof. Reproduce the proof of Proposition 2.1 of [19]. □

Remark 2.3. *On the deterministic case, after an easy calculation (reproduce the proof of Corollary 2.1 of [19]) one can show that the expression of f_0 reduces to*

$$f_0(s):=\inf\left\{\int_{\hat{Y}\setminus D}\widehat{f^{\infty,p}}(\hat{\nabla}w(x))\,dx\ :\ w\in Adm_{\#}(s)\right\}$$

$$Adm_{\#}(s):=\left\{w\in W^{1,p}_{\#}(\hat{Y},\mathbb{R}^3)\ :\ \fint_{\hat{Y}}w\,dx=s,\ w=0\ \text{ on }\hat{Y}\cap D\right\}$$

where $W^{1,p}_{\#}(\hat{Y},\mathbb{R}^3)$ is the set of $\hat{Y}$-periodic functions in $W^{1,p}(\mathcal{O},\mathbb{R}^3)$.

We end this section by the following proposition which is a consequence of Theorem 2.1 when $\mathcal{S}$ is additive. It extends the Birkoff ergodic theorem.

Proposition 2.2. *Le n be fixed in $\mathbb{N}^*$, and $\psi:\Omega\times\mathbb{R}^2\longrightarrow\mathbb{R}$ be a $\mathcal{A}\otimes\mathcal{B}(\mathbb{R}^2)$-measurable function satisfying the three conditions:*

i) for $\mathbf{P}$-almost every $\omega\in\Omega$, $\hat{y}\mapsto\psi(\omega,\hat{y})$ belongs to $L^1_{loc}(\mathbb{R}^2)$;

ii) for all bounded Borel set $\hat{A}$ of $\mathbb{R}^2$ the map $\hat{A}\mapsto\int_{\hat{A}}\psi(\omega,\hat{y})\,d\hat{y}$ belongs to $L^1(\Omega,\mathcal{A},\mathbf{P})$;

iii) for all $z\in n\mathbf{Z}^2$, for all $\hat{y}\in\mathbb{R}^2$, $\psi(\omega,\hat{y}+z)=\psi(\tau_z\omega,\hat{y})$ for $\mathbf{P}$-almost every $\omega\in\Omega$.

Then almost surely

$$\psi(\omega,\frac{\cdot}{\varepsilon})\overset{*}{\rightharpoonup}\mathbf{E}\fint_{(0,n)^2}\psi(.,\hat{y})\,d\hat{y}$$

for the $\sigma(L^1(\mathcal{O}),L^\infty(\mathcal{O}))$ topology.

Proof. The assertion is a straightforward consequence of Theorem 4.2 and Proposition 5.3 in [9]. □

Note that the indicatrice function of the random set $T_\varepsilon \cap \mathcal{O}$ may be written $\mathbf{1}_{D(\omega)\cap\widehat{\mathcal{O}}}(\frac{\cdot}{\varepsilon})$ and that $(\omega,\hat{x}) \mapsto \mathbf{1}_{D(\omega)\cap\widehat{\mathcal{O}}}(.)$ satisfies the condition $\mathbf{1}_{D(\omega)\cap\widehat{\mathcal{O}}}(\hat{x}+z) = \mathbf{1}_{D(\tau_z\omega)\cap\widehat{\mathcal{O}}}(\hat{x})$. Therefore, applying Proposition 2.2 we infer that for $\mathbf{P}$ a.e. ω in Ω,

$$\mathbf{1}_{D\cap\widehat{\mathcal{O}}}(\frac{\cdot}{\varepsilon}) \overset{*}{\rightharpoonup} \int_\Omega |\hat{Y}\cap D(\omega)|\, d\mathbf{P}(\omega). \tag{2.7}$$

We will denote the limit $\displaystyle\int_\Omega |\hat{Y}\cap D(\omega)|\, d\mathbf{P}(\omega)$ in (2.7) by θ and call it the *asymptotic volume fraction of the fibers.*

2.4 Main results

In what follows we assume $a>0$, $p>1$, $b \le p-1+\frac{a}{p}$ and set

$$V_0(\mathcal{O},\mathbb{R}^3) := \left\{ v \in L^p(\mathcal{O},\mathbb{R}^3) : \frac{\partial v}{\partial x_3} \in L^p(\mathcal{O},\mathbb{R}^3),\ v(\hat{x},0)=0 \right\}.$$

Lemma 2.1 (Compactness)**.** *Consider a sequence $(u_\varepsilon)_{\varepsilon>0}$ in $L^p(\mathcal{O},\mathbb{R}^3)$ of bounded energy, i.e., satisfying for $\mathbf{P}$ a.s. $\omega\in\Omega$, $\sup_{\varepsilon>0} E_\varepsilon(\omega,u_\varepsilon) < +\infty$. Then, for $\mathbf{P}$ a.s. $\omega\in\Omega$, there exist a subsequence possibly depending on ω and $(u,v) \in L^p(\mathcal{O},\mathbb{R}^3)\times V_0(\mathcal{O},\mathbb{R}^3)$ possibly depending on ω such that :*

$$u_\varepsilon \rightharpoonup u \quad \text{in} \quad L^p(\mathcal{O},\mathbb{R}^3), \quad \text{and} \quad \frac{\partial u}{\partial x_3} = 0; \tag{2.8}$$

$$\varepsilon^{-\gamma} a(\omega,\frac{\cdot}{\varepsilon}) u_\varepsilon \rightharpoonup v \text{ in } L^p(\mathcal{O},\mathbb{R}^3); \tag{2.9}$$

$$\varepsilon^{-\gamma} a(\omega,\frac{\cdot}{\varepsilon}) \frac{\partial u_\varepsilon}{\partial x_3} \rightharpoonup \frac{\partial v}{\partial x_3} \text{ in } L^p(\mathcal{O},\mathbb{R}^3); \tag{2.10}$$

Remark 2.4. *In an unloaded body, If we only look at the free energy of the material, (2.8), (2.9) and (2.10) hold under the only conditions $a>0$ and $p>1$.*

As suggested by the statement of Lemma 2.1, for every fixed $\omega\in\Omega$ we introduce the following convenient notation for any sequence $(u_\varepsilon)_{\varepsilon>0}$ in $L^p(\mathcal{O},\mathbb{R}^3)$:

$$u_\varepsilon \rightharpoonup\!\!\rightharpoonup (u,v) \iff \begin{cases} u_\varepsilon \rightharpoonup u \in L^p(\mathcal{O},\mathbb{R}^3) \\ \\ \varepsilon^{-\gamma} a(\omega,\frac{\cdot}{\varepsilon}) u_\varepsilon \rightharpoonup v \text{ in } L^p(\mathcal{O},\mathbb{R}^3). \end{cases}$$

In a sense made precise below, we are going to establish the almost sure variational convergence of the sequence $(E_\varepsilon(\omega,.))_{\varepsilon>0}$, associated with the convergence $\rightharpoonup\!\!\rightharpoonup$ above, toward a deterministic functional E_0 defined in $L^p(\mathcal{O},\mathbb{R}^3)\times L^p(\mathcal{O},\mathbb{R}^3)$ as follows. Let denote by H_0 the functional defined in $L^p(\mathcal{O},\mathbb{R}^3)\times L^p(\mathcal{O},\mathbb{R}^3)$ by

$$H_0(u,v) = \begin{cases} \displaystyle\int_{\hat{\mathcal{O}}} f_0(u)\, d\hat{x} + \theta^{1-p} \int_{\mathcal{O}} (g^{\infty,p})^\perp(\frac{\partial v}{\partial x_3}) dx \text{ if } (u,v) \in L^p(\mathcal{O},\mathbb{R}^3)\times V_0(\mathcal{O},\mathbb{R}^3) \\ +\infty \text{ otherwise.} \end{cases}$$

where f_0 and $(g^{\infty,p})^\perp$ are the two functions defined in Section 2.3. In the proofs we will consider the two functionals defined in $L^p(\mathcal{O},\mathbb{R}^3)$ by

$$F_0(u) = \int_{\hat{\mathcal{O}}} f_0(u) d\hat{x}$$

and

$$G_0(u) = \begin{cases} \displaystyle\theta^{1-p} \int_{\mathcal{O}} (g^{\infty,p})^\perp(\frac{\partial v}{\partial x_3}) dx \text{ if } v \in V_0(\mathcal{O},\mathbb{R}^3) \\ +\infty \text{ otherwise,} \end{cases}$$

so that $H_0(u,v) = F_0(u) + G_0(v)$ in $L^p(\mathcal{O},\mathbb{R}^3)\times V_0(\mathcal{O},\mathbb{R}^3)$. We define the limit energy E_0 by:

. **Case** (C_1): $b = \gamma$

$$E_0(u,v) = \begin{cases} H_0(u,v) - \displaystyle\int_{\mathcal{O}} L.u \; dx - \int_{\widehat{\mathcal{O}}} l.v \; d\hat{x} \text{ if } (u,v) \in L^p(\mathcal{O},\mathbb{R}^3) \times V_0(\mathcal{O},\mathbb{R}^3) \\ \\ +\infty \text{ otherwise} \end{cases}$$

. **Case** (C_2): $b < \gamma$

$$E_0(u,v) = \begin{cases} H_0(u,v) - \displaystyle\int_{\mathcal{O}} L.u \; dx \text{ if } (u,v) \in L^p(\mathcal{O},\mathbb{R}^3) \times V_0(\mathcal{O},\mathbb{R}^3) \\ \\ +\infty \text{ otherwise.} \end{cases}$$

Here is our main result:

Theorem 2.2. *The sequence of functionals E_ε almost surely converges to the functional E_0 in the following sense: there exists $\Omega' \in \mathcal{A}$ with $\mathbf{P}(\Omega') = 1$ such that for all $\omega \in \Omega'$ one has*

- *i) for all $(u,v) \in L^p(\mathcal{O},\mathbb{R}^3) \times V_0$ and for all sequence $(u_\varepsilon)_{\varepsilon>0}$ in $L^p(\mathcal{O},\mathbb{R}^3)$ such that $u_\varepsilon \rightharpoonup\!\!\rightharpoonup (u,v)$, then $\liminf\limits_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) \geq E_0(u,v)$;*
- *ii) for all $(u,v) \in L^p(\mathcal{O},\mathbb{R}^3) \times V_0$, there exists a sequence $(u_\varepsilon)_{\varepsilon>0}$ in $L^p(\mathcal{O},\mathbb{R}^3)$ such that $u_\varepsilon \rightharpoonup\!\!\rightharpoonup (u,v)$ and $\limsup\limits_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) \leq E_0(u,v)$.*

Corollary 2.1. *Let denote by $\overline{\overline{u_\varepsilon}}(\omega,.)$ the function $x \mapsto \bar{u}_\varepsilon\big(\omega,\hat{x},h(\varepsilon)x_3\big)$, where $\bar{u}_\varepsilon(\omega,.)$ is the solution of $(\mathcal{P}_{H_{\varepsilon,h(\varepsilon)}})$ and assume that $(g^{\infty,p})^\perp$ is differentiable. Then almost surely there exists a subsequence of $(\overline{\overline{u_\varepsilon}}(\omega,.))_{\varepsilon>0}$ such that $\overline{\overline{u_\varepsilon}}(\omega,.) \rightharpoonup \overline{\overline{u}}$ in $L^p(\mathcal{O},\mathbb{R}^3)$ with for a. e. $\hat{x} \in \hat{\mathcal{O}}$,*

$$\overline{\overline{u}}(\hat{x}) \in \partial f_0^*\Big(\bar{L}\Big)$$

where $\bar{L}(\hat{x}) = \displaystyle\int_0^1 L_1(\hat{x},t) \; dt$. Consequently if ∂f_0^* is single valued then almost surely all the sequence $(\overline{\overline{u_\varepsilon}}(\omega,.))_{\varepsilon>0}$ weakly converges in $L^p(\mathcal{O},\mathbb{R}^3)$ to $\overline{\overline{u}}$ defined for a.e. $\hat{x} \in \hat{\mathcal{O}}$ by

$$\overline{\overline{u}}(\hat{x}) = \partial f_0^*\Big(\bar{L}\Big).$$

Moreover almost surely $\varepsilon^{-\gamma}\mathbb{1}_{T_\varepsilon\cap\mathcal{O}}\overline{\overline{u_\varepsilon}}(\omega,.)$ and $\varepsilon^{-\gamma}\mathbb{1}_{T_\varepsilon\cap\mathcal{O}}\frac{\partial\overline{\overline{u_\varepsilon}}(\omega,.)}{\partial x_3}$ weakly converge to $\bar{v}$ and $\frac{\partial\bar{v}}{\partial x_3}$ in $L^p(\mathcal{O},\mathbb{R}^3))$ respectively, where $\bar{v}$ is the unique solution of

$$\begin{cases} -\dfrac{\partial}{\partial x_3}\Big(\dfrac{dg^{\infty,p})^\perp}{ds}(\dfrac{\partial v}{\partial x_3})\Big) = 0 \textit{ in } \mathcal{O}, \\ \\ v(\hat{x},0) = 0 \textit{ on } \widehat{\mathcal{O}}, \\ \\ D(g^{\infty,p})^\perp(\frac{\partial v}{\partial x_3}).e_3 = \theta^{p-1}\tilde{l} \textit{ on } \widehat{\mathcal{O}} + e_3. \end{cases}$$

where $\tilde{l} = \begin{cases} l \textit{ when } b = \gamma \\ 0 \textit{ if } b < \gamma. \end{cases}$

By eliminating the displacement v regarded as an internal variable, from Theorem 2.2 we easily deduce

Corollary 2.2. *The sequence of energies $E_\varepsilon(\omega,.)$ almost surely Γ-converges to the zero-gradient energy functional $\tilde{E}_0(u) := \inf\{E_0(u,v) : v \in V_0\}$ which is explicitely given in $L^p(\widehat{\mathcal{O}},\mathbb{R}^3)$ by*

$$\tilde{E}_0(u) = \begin{cases} \displaystyle\int_{\widehat{\mathcal{O}}} f_0(u) \; d\hat{x} - \int_{\widehat{\mathcal{O}}} u.\bar{L} \; d\hat{x} + G_0(\bar{v}) - \int_{\widehat{\mathcal{O}}} l.\bar{v} \; d\hat{x} \textit{ if } b = \gamma, \\ \displaystyle\int_{\widehat{\mathcal{O}}} f_0(u) \; d\hat{x} - \int_{\widehat{\mathcal{O}}} u.\bar{L} \; d\hat{x} \textit{ if } b < \gamma. \end{cases}$$

3 Proofs of the results

In what follows C will denote various constants which may depend on ω and may vary from line to line.

3.1 Proof of the compactness Lemma

Proof. Fix ω in the subset of Ω of full probability for which (A_1) holds and consider $(u_\varepsilon)_{\varepsilon>0} \in L^p(\mathcal{O}, \mathbb{R}^3)$ such that $\sup_{\varepsilon>0} E_\varepsilon(\omega, u_\varepsilon) < +\infty$. According to the Poincaré-Wirtinger inequality, there exists a constant C such that

$$\begin{aligned}
\int_{\mathcal{O}} \left| u_\varepsilon - \fint_{\mathcal{O}\cap T_\varepsilon} u_\varepsilon \, dx \right|^p dx &\leq C\varepsilon^p \int_{\mathcal{O}} \left| \hat{\nabla} u_\varepsilon \right|^p dx \\
&\leq C\varepsilon^p \int_{\mathcal{O}\setminus T_\varepsilon} \left| \hat{\nabla} u_\varepsilon \right|^p dx + C\varepsilon^{p-a} \int_{\mathcal{O}\cap T_\varepsilon} \left| \hat{\nabla} u_\varepsilon \right|^p dx
\end{aligned}$$

(see [19], Lemma 3.1). By using Poincaré inequality, growth conditions satisfyed by f and g and the fact that $\gamma > 0$, we infer

$$\begin{aligned}
\int_{\mathcal{O}} |u_\varepsilon|^p \, dx &\leq \int_{\mathcal{O}\cap T_\varepsilon} |u_\varepsilon|^p \, dx + C\varepsilon^p \int_{\mathcal{O}\setminus T_\varepsilon} \left| \hat{\nabla} u_\varepsilon \right|^p dx + C\varepsilon^{p-a} \int_{\mathcal{O}\cap T_\varepsilon} \left| \hat{\nabla} u_\varepsilon \right|^p dx \\
&\leq \int_{\mathcal{O}\cap T_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_3} \right|^p dx + C\varepsilon^p \int_{\mathcal{O}\setminus T_\varepsilon} \left| \hat{\nabla} u_\varepsilon \right|^p dx + C\varepsilon^{p-a} \int_{\mathcal{O}\cap T_\varepsilon} \left| \hat{\nabla} u_\varepsilon \right|^p dx \\
&\leq \frac{\varepsilon^{p\gamma}}{\alpha} H_\varepsilon(u_\varepsilon) + C H_\varepsilon(u_\varepsilon) \\
&\leq C H_\varepsilon(u_\varepsilon). \qquad (3.1)
\end{aligned}$$

On the other hand, according to $st \leq \frac{\nu^p}{p} s^p + \frac{1}{q\nu^q} t^q$ with $s \geq 0$, $s \geq 0$ and $\nu > 0$ suitably chosen later, noticing that

$$\int_{\widehat{\mathcal{O}}\cap\varepsilon D} |u_\varepsilon(\hat{x}, 1)|^p \, d\hat{x} \leq \int_{\mathcal{O}\cap T_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_3} \right|^p \, dx,$$

and since $b \leq \gamma$, we deduce

$$\begin{aligned}
H_\varepsilon(\omega, u_\varepsilon) &\leq C + \left| \int_{\mathcal{O}} L_\varepsilon . u_\varepsilon \, dx \right| + \left| \int_{\widehat{\mathcal{O}}\cap\varepsilon D} \varepsilon^{-b} l . u_\varepsilon \, d\hat{x} \right| \\
&\leq C + \frac{1}{q\nu^q} \int_{\mathcal{O}\setminus T_\varepsilon} |L|^q \, dx + \frac{\nu^p}{p} \int_{\mathcal{O}\setminus T_\varepsilon} |u_\varepsilon|^p \, dx + \frac{1}{q\nu^q} \int_{\widehat{\mathcal{O}}\cap\varepsilon D} |l|^q \, d\hat{x} + \frac{\nu^p}{p} \varepsilon^{-pb} \int_{\mathcal{O}\cap T_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_3} \right|^p \, dx \\
&\leq C + \frac{\nu^p}{p} \int_{\mathcal{O}} |u_\varepsilon|^p \, dx + \frac{\nu^p}{\alpha p} H_\varepsilon(\omega, u_\varepsilon). \qquad (3.2)
\end{aligned}$$

Thus

$$(1 - \frac{\nu^p}{\alpha p}) H_\varepsilon(\omega, u_\varepsilon) \leq C + \frac{\nu^p}{p} \int_{\mathcal{O}} |u_\varepsilon|^p \, dx. \qquad (3.3)$$

Combining (3.3) with (3.1) we infer

$$\int_{\mathcal{O}} |u_\varepsilon|^p \, dx \leq C + C \frac{\nu^p}{\delta(\nu)} \int_{\mathcal{O}\setminus T_\varepsilon} |u_\varepsilon|^p \, dx$$

where $\delta(\nu) := \frac{1}{1 - \frac{\nu^p}{\alpha p}}$. Choosing ν small enough in such a way that $C \frac{\nu^p}{\delta(\nu)} < \frac{1}{2}$ we obtain

$$\int_{\mathcal{O}} |u_\varepsilon|^p \, dx \leq C \qquad (3.4)$$

so that u_ε weakly converges to some u in $L^p(\mathcal{O},\mathbb{R}^3)$. Moreover (3.3), (3.4) yield $H_\varepsilon(\omega,u_\varepsilon)\le C$. Therefore, according to the coercivity assumption on f and g, we infer

$$\begin{aligned}
&\varepsilon^{-p}\int_{\mathcal{O}\setminus T_\varepsilon}\left|\frac{\partial u_\varepsilon}{\partial x_3}\right|^p dx+\varepsilon^{-p\gamma}\int_{\mathcal{O}\cap T_\varepsilon}\left|\frac{\partial u_\varepsilon}{\partial x_3}\right|^p dx\le C; \qquad (3.5)\\
&\varepsilon^{-p\gamma}\int_{\mathcal{O}\cap T_\varepsilon}\left|\frac{\partial u_\varepsilon}{\partial x_3}\right|^p dx\le C;\\
&\varepsilon^{-p\gamma}\int_{\mathcal{O}\cap T_\varepsilon}|u_\varepsilon|^p\,dx\le C.
\end{aligned}$$

from which we easily deduce (2.8), (2.9) and (2.10). □

3.2 Proof of the upper bound in Theorem 2.2

This section is devoted to the establishing of the upper bound (ii) in Theorem 2.2.

Proposition 3.1. *There exists a set $\Omega'\in\mathcal{A}$ of full probability such that for all $(u,v)\in L^p(\mathcal{O},\mathbb{R}^3)\times V_0(\mathcal{O},\mathbb{R}^3)$ and all $\omega\in\Omega'$ there exists a sequence $(u_\varepsilon(\omega))_{\varepsilon>0}$ in $L^p(\mathcal{O},\mathbb{R}^3)$ satisfying*

$$\begin{aligned}
&u_\varepsilon(\omega)\rightharpoonup\rightharpoonup(u,v)\\
&E_0(u,v)\ge\limsup_{\varepsilon\to0}E_\varepsilon(\omega,u_\varepsilon(\omega)).
\end{aligned}$$

Proof. For any sequence $(u_\varepsilon(\omega))_{\varepsilon>0}$ satsifying $u_\varepsilon(\omega)\rightharpoonup\rightharpoonup(u,v)$, the limit

$$\lim_{\varepsilon\to0}\int_{\mathcal{O}}L_\varepsilon.u_e(\omega)\,dx=\begin{cases}\displaystyle\int_{\mathcal{O}}L.u\,dx+\int_{\hat{\mathcal{O}}}lv\,d\hat{x}\text{ when }b=\gamma,\\ \displaystyle\int_{\mathcal{O}}L.u\,dx\text{ when }b<\gamma,\end{cases}$$

is easy to establish and left to the reader. Therefore we are reduced to the prove

$$\begin{aligned}
&u_\varepsilon(\omega)\rightharpoonup\rightharpoonup(u,v)\\
&H_0(u,v)\ge\limsup_{\varepsilon\to0}H_\varepsilon(\omega,u_\varepsilon(\omega)).
\end{aligned}$$

for a suitable sequence $(u_\varepsilon(\omega))_{\varepsilon>0}$. We proceed into three steps.

Step 1. We assume $(u,v)\in\mathcal{C}^1_c(\hat{\mathcal{O}},\mathbb{R}^3)\times\Big(\mathcal{C}^1(\mathcal{O},\mathbb{R}^3)\cap V_0(\mathcal{O},\mathbb{R}^3)\Big)$ and we show that there exists a set Ω' of full probability and, for all $\omega\in\Omega'$, a sequence $(u_\varepsilon(\omega))_{\varepsilon>0}$ in $L^p(\mathcal{O},\mathbb{R}^3)$ such that $u_\varepsilon(\omega)\rightharpoonup\rightharpoonup(u,v)$ and

$$\begin{aligned}
&\lim_{\varepsilon\to0}F_\varepsilon(\omega,u_\varepsilon(\omega))=\int_{\hat{\mathcal{O}}}f_0(u)\,d\hat{x}\\
&\lim_{\varepsilon\to0}G_\varepsilon(\omega,u_\varepsilon(\omega))=G_0(v).
\end{aligned}$$

Let $\eta\in\mathbf{Q}^+$ intended to go to 0 and let $(\hat{Q}_{i,\eta})_{i\in I_\eta}$ be a finite family of pairwise disjoint cubes of size η included in $\hat{\mathcal{O}}$, such that

$$\left|\hat{\mathcal{O}}\setminus\bigcup_{i\in I_\eta}\hat{Q}_{i,\eta}\right|=0.$$

Let $z_\eta:=\displaystyle\sum_{i\in I_\eta}u(\hat{x}_{i,\eta})\mathbb{1}_{\hat{Q}_{i,\eta}}$ where $x_{i,\eta}$ is arbitrarily chosen in $\hat{Q}_{i,\eta}$. Since u is a Lipschitz function on $\hat{\mathcal{O}}$, clearly $z_\eta\to u$ in $L^p(\mathcal{O},\mathbb{R}^3)$ when $\eta\to0$.

For every $i\in I_\eta$, and for fixed $n\in\mathbb{N}^*$, consider $w_{i,n}(\omega,.)\in adm_{n\hat{Y}}(\omega,u(\hat{x}_{i,\eta}))$ and $\xi_{i,n}(\omega,.)\in\mathcal{C}^\infty_c(n\hat{Y}\setminus D(\omega))$ such that

$$\int_{n\hat{Y}\setminus D(\omega)}f^{\infty,p}(\nabla w_{i,n}(\omega,\hat{x}),\xi_{i,n}(\omega,\hat{x}))\,d\hat{x}=\inf\left\{\int_{n\hat{Y}}\widehat{f^{\infty,p}}(\nabla w)\,dy:w\in adm_{n\hat{Y}}(\omega,u(\hat{x}_{i,\eta}))\right\}$$

and extend it on $\mathbb{R}^2$ as follows:

$$\tilde{w}_{i,n}(\omega,\hat{x}) = w_{i,n}(\tau_z\omega,\hat{x}-z) \text{ if } x \in \hat{Y}+z,\ z \in n\mathbf{Z}^2;$$

$$\tilde{\xi}_{i,n}(\omega,\hat{x}) = \xi_{i,n}(\tau_z\omega,\hat{x}-z) \text{ if } x \in \hat{Y}+z,\ z \in n\mathbf{Z}^2.$$

It is easy to check that $\tilde{w}_{i,n}$ and $\tilde{\xi}_{i,n}$ satisfies: $\tilde{w}_{i,n}(\omega,\hat{x}+z) = \tilde{w}_{i,n}(\tau_z\omega,\hat{x})$ and $\tilde{\xi}_{i,n}(\omega,\hat{x}+z) = \tilde{\xi}_{i,n}(\tau_z\omega,\hat{x})$ for all $z \in n\mathbf{Z}$. To shorten notation, we sometime drop the dependance on η and we still denote by $w_{i,n}$ and $\xi_{i,n}$ these two functions. According to Proposition 2.2, we have, almost surely when $\varepsilon \to 0$

$$\begin{aligned} f^{\infty,p}(\nabla w_{i,n}(\omega,\frac{\hat{x}}{\varepsilon}),\xi_{i,n}(\omega,\frac{\hat{x}}{\varepsilon})) &\overset{*}{\rightharpoonup} \mathbf{E}\fint_{n\hat{Y}} f^{\infty,p}(\nabla w_{i,n}(\omega,\hat{x}),\xi_{i,n}(\omega,\hat{x}))\,d\hat{x} \\ &= \mathbf{E}\frac{\mathcal{S}_{(0,n)^2}(\omega,u(\hat{x}_{i,\eta}))}{n^2}, \end{aligned} \tag{3.6}$$

and

$$w_{i,n}(\omega,\frac{\cdot}{\varepsilon}) \rightharpoonup \mathbf{E}\fint_{n\hat{Y}} w_{i,n}(\omega,y)\,dy = u(\hat{x}_{i,\eta}). \tag{3.7}$$

Let $(\theta_{i,\delta})_{i\in I_\eta}$ be a partition of unity associated with $(\hat{Q}_{i,\eta})_{i\in I_\eta}$ with $\theta_{i,\delta} \to \mathbb{1}_{Q_{i,\eta}}$ when $\delta \to 0$ (we omit the dependance on η), and consider the following function in $W^{1,p}(\mathcal{O},\mathbb{R}^3)$:

$$u_{\delta,n,\varepsilon}(\omega,x) = \frac{1}{\theta}\varepsilon^\gamma v + \sum_{i\in I_\eta}\theta_{i,\delta}(\hat{x})\Big[w_{i,n}(\omega,\frac{\hat{x}}{\varepsilon}) + \varepsilon^{p-1}x_3\xi_{i,n}(\omega,\frac{\hat{x}}{\varepsilon})\Big].$$

Clearly $u_{\delta,n,\varepsilon} = \dfrac{1}{\theta}\varepsilon^\gamma v$ on $\mathcal{O}\cap T(\varepsilon)$, and

$$\begin{aligned} &\lim_{\delta\to 0}\lim_{\varepsilon\to 0} u_{\delta,n,\varepsilon}(\omega,.) = z_\eta \text{ weakly in } L^p(\mathcal{O},\mathbb{R}^3) \\ &\lim_{\varepsilon\to 0}\varepsilon^{-\gamma}a(\omega,\tfrac{\cdot}{\varepsilon})u_{\delta,n,\varepsilon}(\omega,.) = v \text{ weakly in } L^p(\mathcal{O},\mathbb{R}^3). \end{aligned} \tag{3.8}$$

Let Ω_0 be the set of full probability made up of all $\omega\in\Omega$ for which $a(\omega,\frac{\cdot}{\varepsilon}) \rightharpoonup \theta$ for the $\sigma(\mathrm{L}^\infty(\mathcal{O}),L^1(\mathcal{O}))$ topology and denote by $\Omega_{i,\eta,n}$ the set of full probability made up of all $\omega\in\Omega$ for which (3.6) and (3.7) hold. In what follows we denote the set of full probability $\bigcap_{n\in\mathbb{N}^*}\bigcap_{\eta\in\mathbf{Q}^+}\bigcap_{i\in I_\eta}\Omega_{i,\eta,n}\cap\Omega_0$ by Ω' and we fix $\omega\in\Omega'$.

Now we are going to estimate $F_\varepsilon(\omega,u_{\delta,n,\varepsilon}(\omega,.))$ and $G_\varepsilon(\omega,u_{n,\delta,\varepsilon}(\omega,.))$. For shortened notation we do not indicate the dependance on ω. On $\mathcal{O}\setminus T_\varepsilon$ we have

$$\begin{aligned} \varepsilon\hat{\nabla}u_{\delta,n,\varepsilon}(x) &= \frac{1}{\theta}\varepsilon^{\gamma+1}\hat{\nabla}v(x) + \sum_{i\in I_\eta}\theta_{i,\delta}(\hat{x})\Big[\hat{\nabla}w_{i,n}(\omega,\frac{\hat{x}}{\varepsilon}) + \varepsilon^{p-1}x_3\hat{\nabla}\xi_{i,n}(\omega,\frac{\hat{x}}{\varepsilon})\Big] \\ &+ \sum_{i\in I_\eta}\varepsilon\hat{\nabla}\theta_{i,\delta}(\hat{x})\Big[w_{i,n}(\omega,\frac{\hat{x}}{\varepsilon}) + \varepsilon^{p-1}x_3\xi_{i,n}(\omega,\frac{\hat{x}}{\varepsilon})\Big] \\ &= O(\varepsilon) + \sum_{i\in I_\eta}\theta_{i,\delta}(\hat{x})\hat{\nabla}w_{i,n}(\omega,\frac{\hat{x}}{\varepsilon}) \end{aligned}$$

and

$$\begin{aligned} \varepsilon^{1-p}\frac{\partial u_{\delta,n,\varepsilon}}{\partial x_3}(x) &= \frac{1}{\theta}\varepsilon^{\gamma+1-p}\frac{\partial v}{\partial x_3}(x) + \sum_{i\in I_\eta}\theta_{i,\delta}(\hat{x})\xi_{i,n}(\omega,\frac{\hat{x}}{\varepsilon}) \\ &= O(\varepsilon) + \sum_{i\in I_\eta}\theta_{i,\delta}(\hat{x})\xi_{i,n}(\omega,\frac{\hat{x}}{\varepsilon}) \end{aligned}$$

where $O(\varepsilon)$ may depend on η, n, and δ. Consequently from (2.3), (2.5), (3.6)

$$\begin{aligned}
\lim_{\varepsilon\to 0} F_\varepsilon(\omega, u_{\delta,n,\varepsilon}) &= \lim_{\varepsilon\to 0} \varepsilon^p \int_{\mathcal{O}\setminus T_\varepsilon(\omega)} f(\hat{\nabla} u_{\delta,n,\varepsilon}, \varepsilon^{-p} \frac{\partial u_{\delta,n,\varepsilon}}{\partial x_3})\ dx \\
&= \lim_{\varepsilon\to 0} \int_{\mathcal{O}\setminus T_\varepsilon(\omega)} f^{\infty,p}(\varepsilon \hat{\nabla} u_{\delta,n,\varepsilon}, \varepsilon^{1-p} \frac{\partial u_{\delta,n,\varepsilon}}{\partial x_3})\ dx \\
&= \lim_{\varepsilon\to 0} \sum_{i\in I_\eta} \int_{\hat{Q}_{i,\eta}} \theta_{i,\delta}^p f^{\infty,p}(\hat{\nabla} w_{i,n}(\omega, \frac{\hat{x}}{\varepsilon}), \xi_{i,n}(\frac{\hat{x}}{\varepsilon}))\ d\hat{x} \\
&= \sum_{i\in I_\eta} \int_{\hat{Q}_{i,\eta}} \theta_{i,\delta}^p \mathbf{E} \frac{\mathcal{S}_{(0,n)^2}(\omega, u(\hat{x}_{i,\eta}))}{n^2}\ d\hat{x}.
\end{aligned}$$

Thus, according to Theorem 2.1

$$\begin{aligned}
\lim_{\delta\to 0} \lim_{n\to\infty} \lim_{\varepsilon\to 0} F_\varepsilon(\omega, u_{\delta,n,\varepsilon}) &= \sum_{i\in I_\eta} |\hat{Q}_{i,\eta}| f_0(u(\hat{x}_{i,\eta})) \\
&= \int_{\mathcal{O}} f_0(z_\eta)\ d\hat{x}.
\end{aligned}$$

Finally, letting $\eta \to 0$, we infer

$$\lim_{\eta\to 0} \lim_{\delta\to 0} \lim_{n\to\infty} \lim_{\varepsilon\to 0} F_\varepsilon(\omega, u_{\delta,n,\varepsilon}) = \int_{\mathcal{O}} f_0(u)\ dx. \tag{3.9}$$

The same kind of computation gives (recall that $\gamma = p-1+\frac{a}{p}$ and that $g^{\infty,p}$ is positively homogeneous of degree p)

$$\begin{aligned}
\lim_{\varepsilon\to 0} G_\varepsilon(\omega, u_{\delta,n,\varepsilon}) &= \lim_{\varepsilon\to 0} \varepsilon^{p-a} \int_{\mathcal{O}\cap T_\varepsilon} g(\varepsilon^\gamma \frac{1}{\theta} \hat{\nabla} v, \varepsilon^{\gamma-p} \frac{1}{\theta} \frac{\partial v}{\partial x_3})\ dx \\
&= \lim_{\varepsilon\to 0} \int_{\mathcal{O}\cap T_\varepsilon} g^{\infty,p}(\varepsilon^p \frac{1}{\theta} \hat{\nabla} v, \frac{1}{\theta} \frac{\partial v}{\partial x_3})\ dx \\
&= \theta \int_{\mathcal{O}} g^{\infty,p}(0, \frac{1}{\theta} \frac{\partial v}{\partial x_3})\ dx = G_0(v).
\end{aligned} \tag{3.10}$$

Combining (3.8), (3.9), (3.10) and a standard diagonalization argument[1] furnishes a map $\varepsilon \mapsto (\eta(\varepsilon), \delta(\varepsilon), n(\varepsilon))$ such that

$$\begin{aligned}
&u_\varepsilon(\omega, .) := u_{\eta(\varepsilon),\delta(\varepsilon),n(\varepsilon),\varepsilon}(\omega, .) \rightharpoonup\!\!\rightharpoonup (u, v) \\
&\lim_{\varepsilon\to 0} H_\varepsilon(\omega, u_\varepsilon(\omega, .)) = \int_{\hat{\mathcal{O}}} f_0(u(\hat{x}))\ d\hat{x} + G_0(v).
\end{aligned}$$

which completes the proof of step 1.

Step 2. We fix $(u, v) \in L^p(\mathcal{O}, \mathbb{R}^3) \times V_0(\mathcal{O}, \mathbb{R}^3)$ with $v \in \mathcal{C}^1(\mathcal{O}, \mathbb{R}^3)$ and we show that for all $\omega \in \Omega''$ there exists $(u_\varepsilon(\omega))_{\varepsilon>0}$ in $L^p(\mathcal{O}, \mathbb{R}^3)$ such that $u_\varepsilon(\omega) \rightharpoonup\!\!\rightharpoonup (u, v)$ and $\lim_{\varepsilon\to 0} H_\varepsilon(\omega, u_\varepsilon(\omega)) = H_0(u, v)$. Consider $u_n \in \mathcal{C}_c^1(\hat{\mathcal{O}}, \mathbb{R}^3)$ weakly converging toward u in $\mathrm{L}^p(\hat{\mathcal{O}}, \mathbb{R}^3)$ such that

$$\lim_{n\to+\infty} \int_{\hat{\mathcal{O}}} f_0(u_n)\ d\hat{x} = \int_{\hat{\mathcal{O}}} f_0(u)\ d\hat{x}.$$

Thus according to step 1, there exists $u_{\varepsilon,n}(\omega, .)$ weakly converging to u_n when $\varepsilon \to 0$ and

$$\lim_{n\to n} \lim_{\varepsilon\to 0} H_\varepsilon(\omega, u_{\varepsilon,n}(\omega, .)) = \int_{\hat{\mathcal{O}}} f_0(u(\hat{x}))\ d\hat{x} + G_0(v).$$

[1] One can easily check that $u_{\eta,\delta,n,\varepsilon}(\omega, .)$ and $\varepsilon^{-\gamma} a(\omega, \frac{\cdot}{\varepsilon}) u_{\eta,\delta,n,\varepsilon}$ belongs to a fixed ball $\mathcal{B}(0, r)$ of $L^p(\mathcal{O}, \mathbb{R}^3)$. Since the weak topology of $L^p(\mathcal{O}, \mathbb{R}^3)$ induces a metric on bounded sets, the diagonalization argument holds.

We conclude by a diagonalization argument.

Step 3. For any $(u,v) \in L^p(\mathcal{O},\mathbb{R}^3) \times V_0(\mathcal{O},\mathbb{R}^3)$ we show that for all $\omega \in \Omega''$ there exists $(u_\varepsilon(\omega))_{\varepsilon>0}$ in $L^p(\mathcal{O},\mathbb{R}^3)$ such that $u_\varepsilon(\omega) \rightharpoonup\rightharpoonup (u,v)$ and $\lim_{\varepsilon\to 0} H_\varepsilon(\omega, u_\varepsilon(\omega)) = H_0(u,v)$.

let $v \in V_0(\mathcal{O},\mathbb{R}^3)$. According to standard relaxation results, there exists a sequence $(\zeta_n)_{n\in\mathbb{N}}$ in $\mathcal{C}^1_c(\mathcal{O})$ weakly converging to $\dfrac{\partial v}{\partial x_3}$ in $L^p(\mathcal{O},\mathbb{R}^3)$ such that

$$\lim_{n\to+\infty} \int_{\mathcal{O}} (g^{\infty,p})^\perp(\frac{1}{\theta}\zeta_n) = \int_{\mathcal{O}} (g^{\infty,p})^\perp(\frac{1}{\theta}\frac{\partial v}{\partial x_3})dx. \tag{3.11}$$

For all $x \in \mathcal{O}$ set

$$v_n(x) := \int_0^{x_3} \zeta_n(\hat{x}, s)\ ds.$$

Then $v_n \in V_0(\mathcal{O},\mathbb{R}^3) \cap \mathcal{C}^1(\mathcal{O})$, $v_n \rightharpoonup u$ in $L^p(\mathcal{O},\mathbb{R}^3)$ and

$$\lim_{n\to+\infty} \theta \int_{\mathcal{O}} (g^{\infty,p})^\perp(\frac{1}{\theta}\frac{\partial v_n}{\partial x_3}) = G_0(v).$$

We end the proof by using Step 2 and a diagonalization argument. □

3.3 Proof of the lower bound in Theorem 2.2

This section is devoted to the establishing of the lower bound (i) of Theorem 2.2.

Proposition 3.2. *For all sequence $(u_\varepsilon)_{\varepsilon>0}$ such that $u_\varepsilon \rightharpoonup\rightharpoonup (u,v)$ one has*

$$E_0(u,v) \quad \leq \quad \liminf_{\varepsilon\to 0}\ E_\varepsilon(\omega, u_\varepsilon) \tag{3.12}$$

for $\mathbf{P}$ *a. s.* $\omega \in \Omega$.

Proof. One may assume $\liminf_{\varepsilon\to 0} E_\varepsilon(\omega,u_\varepsilon) < +\infty$ otherwise there is nothing to prove. It suffices to show that for $\mathbf{P}$ a.s. ω in Ω,

$$F_0(v) \quad \leq \quad \liminf_{\varepsilon\to 0}\ F_\varepsilon(\omega, u_\varepsilon) \tag{3.13}$$

$$G_0(u) \quad \leq \quad \liminf_{\varepsilon\to 0}\ G_\varepsilon(\omega, u_\varepsilon). \tag{3.14}$$

Indeed, according to Lemma 2.1, we easily infer

$$\lim_{\varepsilon\to 0}\int_{\mathcal{O}} L_\varepsilon.u_e(\omega)\ dx = \begin{cases} \displaystyle\int_{\mathcal{O}} L.u\ dx + \int_{\widehat{\mathcal{O}}} l.v\ d\hat{x} \text{ when } b=\gamma, \\ \displaystyle\int_{\mathcal{O}} L.u\ dx \text{ when } b<\gamma, \end{cases}$$

Proof of (3.13). Note that since $\sup_{\varepsilon>0} E_\varepsilon(\omega,u_\varepsilon) < +\infty$, from (3.5), we infer that there exists a constante C such that

$$\varepsilon^p \int_{\mathcal{O}\cap T_\varepsilon} \left|\hat{\nabla} u_\varepsilon\right|^p dx < C\varepsilon^a, \tag{3.15}$$

$$\int_{\mathcal{O}\cap T_\varepsilon} \left|\frac{\partial u_\varepsilon}{\partial x_3}\right|^p < C\varepsilon^{p\gamma}. \tag{3.16}$$

On the other hand, according to the compactness lemma, Lemma 2.1, one has for a subsequence $\mathbf{1}_{\mathcal{O}\cap T_\varepsilon} u_\varepsilon \to 0$ in $L^p(\mathcal{O},\mathbb{R}^3)$ so that,

$$\mathbf{1}_{\mathcal{O}\setminus T_\varepsilon} u_\varepsilon = u_\varepsilon - \mathbf{1}_{\mathcal{O}\cap T_\varepsilon} u_\varepsilon \rightharpoonup u \text{ in } L^p(\mathcal{O},\mathbb{R}^3). \tag{3.17}$$

We will make use of (3.17) in the last step of the proof.

From (2.3), the coercivity condition satisfied by f and g, and from (3.15), (3.15), it is easily seen that

$$\begin{aligned}\liminf_{\varepsilon\to 0}\varepsilon^p\int_{\mathcal{O}\setminus T_\varepsilon} f(\hat{\nabla}u_\varepsilon,\frac{1}{\varepsilon^p}\frac{\partial u_\varepsilon}{\partial x_3})dx &= \liminf_{\varepsilon\to 0}\int_{\mathcal{O}}\varepsilon^p f(\hat{\nabla}u_\varepsilon,\frac{1}{\varepsilon^p}\frac{\partial u_\varepsilon}{\partial x_3})dx\\ &\geq \liminf_{\varepsilon\to 0}\int_{\mathcal{O}}\widehat{f^{\infty,p}}(\varepsilon\hat{\nabla}u_\varepsilon)dx\end{aligned}$$

Fix x_0 in $\mathcal{O}$ and set $Q_\rho(x_0):=S_\rho(\hat{x}_0)\times I_\rho(x_{0,3})$ (to shorten notation we sometimes do not indicate the fixed argument x_0). By using a blow up argument, for proving 3.13, it is enough to establish that for a.e. x_0 in $\mathcal{O}$,

$$\lim_{\rho\to 0}\liminf_{\varepsilon\to 0}⨍_{Q_\rho(x_0)}\widehat{f^{\infty,p}}(\varepsilon\nabla u_\varepsilon)\ dx\geq f_0^{**}(u(x_0)).\tag{3.18}$$

Let $0<\delta<1$ intended to go to 1 and set $(T_\varepsilon)_\delta=\varepsilon D_\delta(\omega)\times(0,1)$ where $D_\delta(\omega)=\bigcup_{i\in\mathbb{N}}(\omega_i+\hat{B}_{\delta\frac{d}{2}}(0)$. Let denote by $\hat{A}\mapsto\mathcal{S}_{\hat{A}}(\omega,s,\delta)$ the subadditive process introduced in Section 2.3 where $D(\omega_i)$ is replaced by the disk $D_\delta(\omega_i):=\omega_i+\hat{B}_{\delta\frac{d}{2}}(0)$ and denote by $\mathrm{Adm}_{\hat{A}}(\omega,s,\delta)$ the associated admissible set. Denoting by $C_{\varepsilon,\rho}$ be the smallest cube in $\mathcal{I}$ containing $\frac{1}{\varepsilon}S_\rho$, our strategy consists in suitably changing the function u_ε in order to obtain a function z_ε whose mean $⨍_{I_\rho}z(\hat{x},x_3)\ dx_3$ belongs to $\mathrm{Adm}_{C_{\varepsilon,\rho}}(\omega,u(x_0),\delta)$ and whose gradient decreases the left hand side of (3.18). In the four steps below, to shorten notation, we do not indicate the dependance on ρ for the various Sobolev functions.

First change. By using a standard truncation argument, we modify u_ε into a Sobolev function satisfying $u_{\varepsilon,\delta}=0$ in $(T_\varepsilon)_\delta$ and

$$⨍_{Q_\rho}\mathbf{1}_{Q_\rho\setminus T_\varepsilon}f^{\infty,p}(\varepsilon\nabla u_\varepsilon)\ dx\geq⨍_{Q_\rho}f^{\infty,p}(\varepsilon\nabla u_{\varepsilon,\delta})\ dx-C\frac{\varepsilon}{(1-\delta)^p}\tag{3.19}$$

Indeed, consider φ in $\mathcal{C}_c^1(S_\rho)$ satisfying $\varphi=0$ in εD_δ, $\varphi=1$ in $S_\rho\setminus\varepsilon D$ and $|\nabla\varphi|_\infty\leq\frac{1}{\varepsilon(1-\delta)}$ and set

$$u_{\varepsilon,\delta}=\varphi u_\varepsilon.$$

According to the growth conditions satisfied by $f^{\infty,p}$ we infer

$$\begin{aligned}\int_{Q_\rho}\widehat{f^{\infty,p}}(\varepsilon\hat{\nabla}u_{\varepsilon,\delta})\ d\hat{x} &= \int_{Q_\rho\setminus T_\varepsilon}\widehat{f^{\infty,p}}(\varepsilon\hat{\nabla}u_\varepsilon)\ d\hat{x}+\int_{(T_\varepsilon\setminus(T_\varepsilon)_\delta)\cap Q_\rho}\widehat{f^{\infty,p}}(\varepsilon\hat{\nabla}u_{\varepsilon,\delta})\ d\hat{x}\\ &\leq \int_{Q_\rho}\widehat{f^{\infty,p}}(\varepsilon\hat{\nabla}u_\varepsilon)\ d\hat{x}+\beta\int_{(T_\varepsilon\setminus(T_\varepsilon)_\delta)\cap Q_\rho}\varepsilon^p\left|\hat{\nabla}u_\varepsilon\right|^p\ d\hat{x}\\ &\quad+\beta\frac{1}{(1-\delta)^p}\int_{(T_\varepsilon\setminus(T_\varepsilon)_\delta)\cap Q_\rho}|u_\varepsilon|^p\ d\hat{x}\end{aligned}$$

so that, from the Poincaré inequality and (3.15), (3.16), we infer

$$\int_{Q_\rho}\varepsilon^p\widehat{f^{\infty,p}}(\hat{\nabla}u_{\varepsilon,\delta})\ d\hat{x}\leq\int_{Q_\rho}\varepsilon^p\widehat{f^{\infty,p}}(\hat{\nabla}u_\varepsilon)\ d\hat{x}+\beta\left[\frac{\varepsilon^{p\gamma}}{(1-\delta)^p}+\varepsilon^a\right]$$

which proves (3.19).

Second change. By using a standard De Giorgi slicing argument (see for instance [4], proof of Proposition 11.2.3), there exists $\eta(\varepsilon)\to 0^+$, $\eta(\varepsilon)>\varepsilon$, a $\eta(\varepsilon)$-neighbohrood $V\eta(\varepsilon)\subset Q_\rho$ of ∂Q_ρ, and a Sobolev function $\tilde{u}_{\varepsilon,\delta}$ vanishing on $\partial S_\rho\times I_\rho$, equal to $u_{\varepsilon,\delta}$ in a $Q_\rho\setminus V\eta(\varepsilon)$, satisfying

$$⨍_{Q_\rho}f^{\infty,p}(\varepsilon\hat{\nabla}u_{\varepsilon,\delta})\ dx\geq⨍_{Q_\rho}f^{\infty,p}(\varepsilon\hat{\nabla}\tilde{u}_{\varepsilon,\delta})\ dx-\frac{C(\rho)}{\nu}-r_\varepsilon(\rho)\tag{3.20}$$

where $C(\rho)$ is a positive constant depending only on ρ, $\lim_{\varepsilon\to 0} r_\varepsilon(\rho) = 0$ and $\nu \in \mathbb{N}$ is the number of bands slicing $V\eta(\varepsilon)$ and intended to go to $+\infty$. It is worth noticing that $\tilde{u}_{\varepsilon,\delta}$ remains equal to 0 in $(T_\varepsilon)_\delta$ since it is of the form $\varphi_{\eta(\varepsilon)} u_{\varepsilon,\delta}$ for a suitable truncation function $\varphi_{\eta(\varepsilon)}$.

Third change. We modify $\tilde{u}_{\varepsilon,\delta}$ into a Sobolev function $w_{\varepsilon,\delta}$ satisfying

$$w_{\varepsilon,\delta} = 0 \text{ in } (T_\varepsilon)_\delta, \quad w_{\varepsilon,\delta} = 0 \text{ on } \partial Q_\rho, \quad ⨍_{Q_\rho} w_{\varepsilon,\delta} = u(x_0)$$

and

$$⨍_{Q_\rho} f^{\infty,p}(\varepsilon \hat{\nabla} \tilde{u}_{\varepsilon,\delta})\, dx \geq ⨍_{Q_\rho} f^{\infty,p}(\varepsilon \hat{\nabla} w_{\varepsilon,\delta})\, dx - C\left| u(x_0) - ⨍_{Q_\rho} \mathbf{1}_{Q_\rho \setminus (T_\varepsilon)_\delta} \tilde{u}_{\varepsilon,\delta}\, dy \right|^p. \tag{3.21}$$

Indeed, set

$$w_{\varepsilon,\delta} = \tilde{u}_{\varepsilon,\delta} + \frac{\psi}{⨍_{Q_\rho} \psi\, dx}\Big(u(x_0) - ⨍_{Q_\rho} \mathbf{1}_{Q_\rho \setminus (T_\varepsilon)_\delta} \tilde{u}_{\varepsilon,\delta}\, dy \Big)$$

where $\psi \in \mathcal{C}^1_c(Q_\rho)$ satisfies $\psi = 0$ in T_ε, $\psi = 0$ on ∂Q_ρ, $|\nabla \psi|_\infty \leq \frac{C}{\varepsilon}$ and $|\psi|_\infty \leq C$.

Last step. Collecting (3.19), (3.20) and (3.21) we finally obtain

$$\begin{aligned} ⨍_{Q_\rho} f^{\infty,p}(\varepsilon \hat{\nabla} u_\varepsilon)\, dx \quad &\geq \quad ⨍_{Q_\rho} f^{\infty,p}(\varepsilon \hat{\nabla} w_{\varepsilon,\delta})\, dx - C\frac{\varepsilon}{(1-\delta)^p} - \frac{C(\rho)}{\nu} - r_\varepsilon(\rho) \\ &\qquad -C\left| u(x_0) - ⨍_{Q_\rho} \mathbf{1}_{Q_\rho \setminus (T_\varepsilon)_\delta} \tilde{u}_{\varepsilon,\delta}\, dy \right|^p. \end{aligned}$$

Set $z_{\varepsilon,\delta}(y) := w_{\varepsilon,\delta}(\varepsilon y)$. A change of scale then yields

$$\begin{aligned} ⨍_{Q_\rho} f^{\infty,p}(\varepsilon \hat{\nabla} u_\varepsilon)\, dx \quad &\geq \quad ⨍_{\frac{1}{\varepsilon}Q_\rho} f^{\infty,p}(\hat{\nabla} z_{\varepsilon,\delta})\, dx - C\frac{\varepsilon}{(1-\delta)^p} - \frac{C(\rho)}{\nu} - r_\varepsilon(\rho) \\ &\qquad -C\left| u(x_0) - ⨍_{Q_\rho} \mathbf{1}_{Q_\rho \setminus (T_\varepsilon)_\delta} \tilde{u}_{\varepsilon,\delta}\, dy \right|^p. \end{aligned}$$

Extend $z_{\varepsilon,\delta}$ by 0 in $\mathbb{R}^3 \setminus \frac{1}{\varepsilon} S_\rho$. Then the function $\tilde{z}_{\varepsilon,\delta}$ defined by

$$\tilde{z}_{\varepsilon,\delta}(\hat{x}) := \frac{|C_{\varepsilon,\rho}|}{|\frac{1}{\varepsilon} S_\rho|} ⨍_{I_\rho} z_{\varepsilon,\delta}(\hat{x}, x_3)\, dx_3$$

clearly belongs to $\mathrm{Adm}_{C_{\varepsilon,\rho}}(\omega, u(x_0), \delta)$. Therefore, according to Jensen's inequality and from the p-homogeneity of $f^{\infty,p}$

$$\begin{aligned} ⨍_{Q_\rho} f^{\infty,p}(\varepsilon \hat{\nabla} u_\varepsilon)\, dx \quad &\geq \quad \Big(\frac{|C_{\varepsilon,\rho}|}{|\frac{1}{\varepsilon} S_\rho|}\Big)^p \frac{\mathcal{S}_{C_{\varepsilon,\rho}}(\omega, u(x_0), \delta)}{|C_{\varepsilon,\rho}|} - C\frac{\varepsilon}{(1-\delta)^p} - \frac{C(\rho)}{\nu} - r_\varepsilon(\rho) \\ &\qquad -C\left| u(x_0) - ⨍_{Q_\rho} \mathbf{1}_{Q_\rho \setminus (T_\varepsilon)_\delta} \tilde{u}_{\varepsilon,\delta}\, dy \right|^p. \end{aligned} \tag{3.22}$$

It is easily seen that from (3.17) and the Lebesgue point Theorem, for a.e. x_0 in $\mathcal{O}$ one has

$$\lim_{\rho\to 0} \lim_{\delta\to 1} \lim_{\varepsilon\to 0} \left| u(x_0) - ⨍_{Q_\rho} \mathbf{1}_{Q_\rho \setminus (T_\varepsilon)_\delta} \tilde{u}_{\varepsilon,\delta}\, dy \right|^p = 0.$$

Moreover $\lim_{\varepsilon\to 0} \frac{|C_{\varepsilon,\rho}|}{|\frac{1}{\varepsilon} S_\rho|} = 1$. On the other hand, according to Theorem 2.1, Theorem 5.2 and Section 6.2 of [18], for $\mathbf{P}$-almost every $\omega \in \Omega$ and for every $\rho > 0$ one has

$$\lim_{\delta\to 1} \lim_{\varepsilon\to 0} \frac{\mathcal{S}_{C_{\varepsilon,\rho}}(\omega, u(x_0), \delta)}{|C_{\varepsilon,\rho}|} = \lim_{\varepsilon\to 0} \frac{\mathcal{S}_{C_{\varepsilon,\rho}}(\omega, u(x_0))}{|C_{\varepsilon,\rho}|} = f_0(u(x_0)). \tag{3.23}$$

Then, letting successively $\varepsilon \to 0$, $\delta \to 1$, $\nu \to \infty$ and $\rho \to 0$ in (3.22), we obtain for $\mathbf{P}$-almost every $\omega \in \Omega$ and for almost every $x_0 \in \mathcal{O}$,

$$\liminf_{\varepsilon\to 0} \fint_{Q_\rho} f^{\infty,p}(\varepsilon \nabla u_\varepsilon)\ dx \geq f_0(u(x_0))$$

which ends the proof.

Proof of (3.14). Fix ω in the set Ω" of full probability given in Proposition 2.2 and assume that $\liminf_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon) < +\infty$. According to the Moreau-Rockafellar duality principle we infer that for all ϕ in $L^q(\mathcal{O})$:

$$\begin{aligned}
\liminf_{\varepsilon\to 0}\ G_\varepsilon(u_\varepsilon) &= \liminf_{\varepsilon\to 0} \varepsilon^{p-a} \int_{\mathcal{O}} a(\omega, \frac{\hat{x}}{\varepsilon}) g^{\infty,p}(\hat{\nabla} u_\varepsilon, \frac{1}{\varepsilon^p}\frac{\partial u_\varepsilon}{\partial x})dx \\
&\geq \liminf_{\varepsilon\to 0} \int_{\mathcal{O}} a(\omega, \frac{\hat{x}}{\varepsilon})(g^{\infty,p})^{\perp}(\varepsilon^{-\gamma}\frac{\partial u_\varepsilon}{\partial x_3})dx \\
&\geq \liminf_{\varepsilon\to 0} \Big(\int_{\mathcal{O}} a(\omega, \frac{\hat{x}}{\varepsilon})\phi.\varepsilon^{-\gamma}\frac{\partial u_\varepsilon}{\partial x_3}dx - \int_{\mathcal{O}} a(\omega, \frac{\hat{x}}{\varepsilon})(g^{\infty,p})^{\perp,*}(\phi)dx \Big) \\
&= \int_{\mathcal{O}} \phi.\frac{\partial v}{\partial x_3}dx - \theta \int_{\mathcal{O}} (g^{\infty,p})^{\perp,*}(\phi)dx \\
&= \theta\ \left[\int_{\mathcal{O}} \frac{1}{\theta}\phi\frac{\partial v}{\partial x_3}dx - \int_{\mathcal{O}} (g^{\infty,p})^{\perp,*}(\phi)dx \right].
\end{aligned}$$

By taking the the supremum over all functions ϕ in $\phi \in L^q(\mathcal{O}, \mathbb{R}^3)$ we finally obtain

$$\begin{aligned}
\liminf_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon) &\geq \theta \sup_{\phi\in L^q(\mathcal{O})} \left[\int_{\mathcal{O}} \frac{1}{\theta}\phi\frac{\partial v}{\partial x_3}dx - \int_{\mathcal{O}} (g^{\infty,p})^{\perp,*}(\phi)dx \right] \\
&= \theta \int_{\mathcal{O}} (g^{\infty,p})^{\perp}(\frac{1}{\theta}\frac{\partial v}{\partial x_3})dx = \theta^{1-p} \int_{\mathcal{O}} (g^{\infty,p})^{\perp}(\frac{\partial v}{\partial x_3})dx
\end{aligned}$$

which completes the proof. $\square$

3.4 Proof of Corollary 2.1

Apply the variational property of the convergence established in Theorem 2.2, then compute the Euler equation associated with the minimization problem $\min\left\{E_0(u,v) : (u,v) \in L^p(\mathcal{O}, \mathbb{R}^3) \times V_0(\mathcal{O}, \mathbb{R}^3)\right\}$ to obtain:

$$\begin{cases}
\partial f_0(\overline{\overline{u}}(\hat{x})) \ni \displaystyle\int_0^1 L(\hat{x}, s)\ ds, \\
-\dfrac{\partial}{\partial x_3}\Big(\dfrac{d g^{\infty,p})^{\perp}}{ds}(\dfrac{\partial v}{\partial x_3})\Big) = 0 \text{ in } \mathcal{O}, \\
v(\hat{x}, 0) = 0 \text{ on } \widehat{\mathcal{O}}, \\
D(g^{\infty,p})^{\perp}(\frac{\partial v}{\partial x_3}).e_3 = \theta^{p-1}\tilde{l} \text{ on } \widehat{\mathcal{O}} + e_3.
\end{cases}$$

To end the proof it suffices to apply the subdifferential rule:

$$a^* \in \partial f_0(a) \iff a \in \partial f_0^*(a^*).$$

3.5 Computation of $\overline{\overline{u}}$ when $f = \frac{1}{2}|.|^2$

In this section we establish the expression of $\overline{\overline{u}}$ when $f = \frac{1}{2}|.|^2$. With the notation of Corollary 2.1 we have:

Proposition 3.3. *Let denote by $U_n(\omega,.)$ the unique solution of the scalar random Dirichlet problem*

$$\begin{cases} -\Delta U = 1 \text{ in } n\hat{Y} \setminus D(\omega), \\ U \in W_0^{1,2}(n\hat{Y} \setminus D(\omega)), \end{cases}$$

and set $\Lambda_n(\omega) := \fint_{n\hat{Y}} U_n(\omega,\hat{x})\, d\hat{x}$. Then for $\mathbf{P}$ a.e. $\omega \in \Omega$, $\Lambda_n(\omega)$ converges to a deterministic value $\Lambda > 0$ and $\overline{\overline{u}}$ is uniquely determined by the formula

$$\overline{\overline{u}}(\hat{x}) = \frac{\Lambda}{2} \int_0^1 L(\hat{x},t)\, dt. \tag{3.24}$$

Proof. Consider the Lagrange multiplier $\lambda^{s,n}(\omega) \in \mathbb{R}^3$ of the optimization problem

$$f_n(\omega,s) := \inf \left\{ \frac{1}{2} \fint_{n\hat{Y}} |\nabla w|^2 \, d\hat{x} : w \in W_0^{1,2}(n\hat{Y} \setminus D(\omega), \mathbb{R}^3),\ \fint_{n\hat{Y}} w\, d\hat{x} = s \right\},$$

whose (random) minimizer w^s satisfies

$$\begin{cases} -\Delta w^s = \lambda^{s,n}(\omega) \text{ in } n\hat{Y} \setminus D(\omega), \\ w^s \in W_0^{1,2}(n\hat{Y} \setminus D(\omega), \mathbb{R}^3), \\ \fint_{n\hat{Y}} w^s\, d\hat{x} = s. \end{cases} \tag{3.25}$$

Let denote by $\mathcal{B} = (e_1,e_2,e_3)$ the canonical basis of $\mathbb{R}^3$. Applying (3.25) for $s = e_i$ we deduce $f_n(\omega,e_i) = \frac{1}{2}\lambda^{e_i,n}(\omega).e_i$. Then, applying (3.25) for $s = e_i + e_j$ and $s = e_i - e_j$, and noticing that $\lambda^{e_i \pm e_j,n}(\omega) = \lambda^{e_i,n}(\omega) \pm \lambda^{e_j,n}(\omega)$, we easily infer

$$\frac{1}{4}\big[f_n(\omega,e_i+e_j) - f_n(\omega,e_i-e_j)\big] = \frac{1}{2}\big[\lambda^{e_i,n}(\omega).e_j + \lambda^{e_j,n}(\omega).e_i\big].$$

The same calculation holds when replacing e_i and e_j by any vector u and v. This proves that $f_n(\omega,.)$ is a quadratic form (note that $f_n(\omega,.)$ is homogeneous of degree 2), and that $f_n(\omega,s) = A^n(\omega)s\ .\ s$ where $A^n(\omega)$ is a symetrical 3×3 matrix given by

$$A^n_{ij}(\omega) = \frac{1}{2}\big[\lambda^{e_i,n}(\omega).e_j + \lambda^{e_j,n}(\omega).e_i\big].$$

Applying (3.25) with $s = e_i$ and taking w^{e_j} as a test function and, symetrically with $s = e_j$ and taking w^{e_i} as a test function, we deduce $\lambda^{e_i,n}(\omega).e_j = \lambda^{e_j,n}(\omega).e_i$, so that $A^n_{ij}(\omega) = \lambda^{e_i,n}(\omega).e_j$. But, for $i \neq j$, from (3.25), we infer that $w^{e_i}.e_j$ satisfies the problem

$$\begin{cases} -\Delta w^{e_i}.e_j = \lambda^{e_i,n}(\omega).e_j \text{ in } n\hat{Y} \setminus D(\omega), \\ w^{e_i}.e_j \in W_0^{1,2}(n\hat{Y} \setminus D(\omega)), \\ \fint_{n\hat{Y}} w^{e_i}.e_j\, d\hat{x} = 0. \end{cases}$$

from which we deduce $w^{e_i}.e_j = 0$ (take $w^{e_i}.e_j$ as a test function), thus $A^n_{ij}(\omega) := \lambda^{e_i,n}(\omega).e_j = 0$ for $i \neq j$.

Let us compute $A^n_{ii}(\omega) = \lambda^{e_i,n}(\omega).e_i$. Since $\lambda^{e_i,n}(\omega).e_i = 2f_n(\omega,e_i)$, one has $\lambda^{e_i,n}(\omega).e_i \geq 2\alpha > 0$ and from (3.25), we infer that $\dfrac{w^{e_i}.e_i}{\lambda^{e_i,n}(\omega).e_i}$ solves the scalar Dirichlet problem

$$\begin{cases} -\Delta U = 1 \text{ in } n\hat{Y} \setminus D(\omega), \\ U \in W_0^{1,2}(n\hat{Y} \setminus D(\omega)). \end{cases}$$

Let us denote by $U_n(\omega)$ its unique solution, then

$$A^n_{ii}(\omega) = \frac{1}{⨍_{n\hat{Y}} U_n(\omega)\, d\hat{x}}.$$

By using once again the subadditive ergodic theorem one can prove that $\Lambda_n(\omega) := ⨍_{n\hat{Y}} U_n(\omega)\, d\hat{x}(\omega)$ almost surely converges to a deterministic value $\Lambda > 0$ (see [10]). Consequently, for **P**- a.e. $\omega \in \Omega$ and for all $s \in \mathbb{R}^3$,

$$\lim_{n\to+\infty} f_n(\omega, s) = f_0(s) = \frac{1}{\Lambda} s.s$$

and $\partial f_0(s) = \frac{2}{\Lambda} s$. The conclusion follows from $\partial f_0^*(s) = \frac{\Lambda}{2} s$.

□

3.6 Numerical result

We would like to compute an approximation $\Lambda_n(\omega)$ of the constant Λ determined in the preceding section. For this we make use of the cast3M program to solve the scalar random Dirichlet problem

$$\begin{cases} -\Delta U = 1 \text{ in } n\hat{Y} \setminus D(\omega), \\ U \in W_0^{1,2}(n\hat{Y} \setminus D(\omega)), \end{cases}$$

in the geometrical situation of the random chessboard-like described in Example 2.1 with Ω_0 made up of 9 points. The first part of the program consists in constructing the random mesh associated with $n\hat{Y} \setminus D(\omega)$ (Fig (2)). The sections of the fibers are randomly placed following 9 different places in each cell.

Figure 3 represents the evolution of $n \mapsto \Lambda_n(\omega)$ for various realizations ω (in case of equi-probability $\alpha_k = \frac{1}{9}$) when n increases. For each ω we can see that the value of $\Lambda_n(\omega)$ converges to the same constant Λ. This illustrates ergodicity. We also compare the various results obtained by varying the probability presence α_0 of the section of the periodic case with the value Λ_n of the periodic case (the curve in red) in Figure 4. Figure 5 represents the evolution of $\Lambda_n(\omega)$ when the radius of the sections increases.

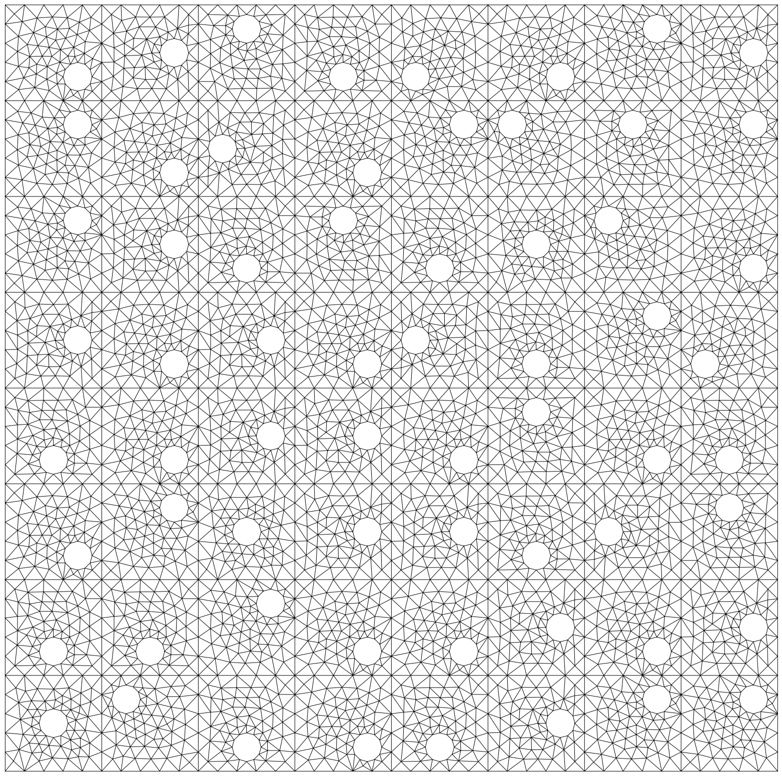

Figure 2: A "chessboard-like" random mesh (n=8).

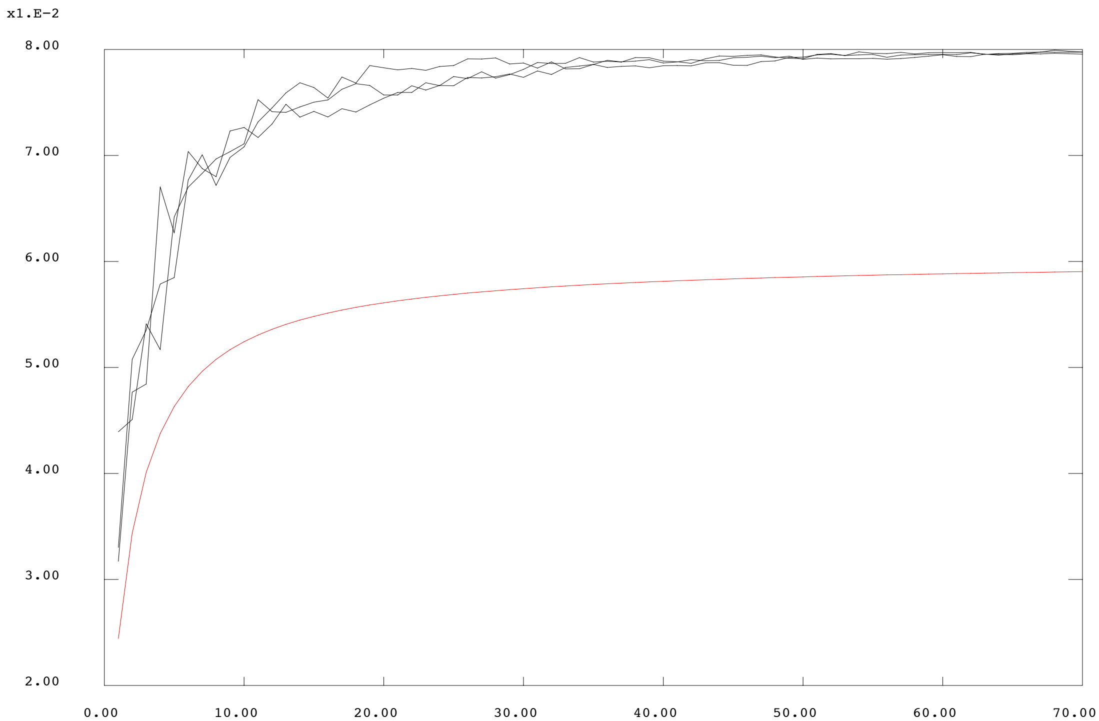

Figure 3: The curves $n \mapsto \Lambda_n(\omega)$ for various realizations ω with equi-probability presence $\alpha_k = \frac{1}{9}$, k=1,…, 9.

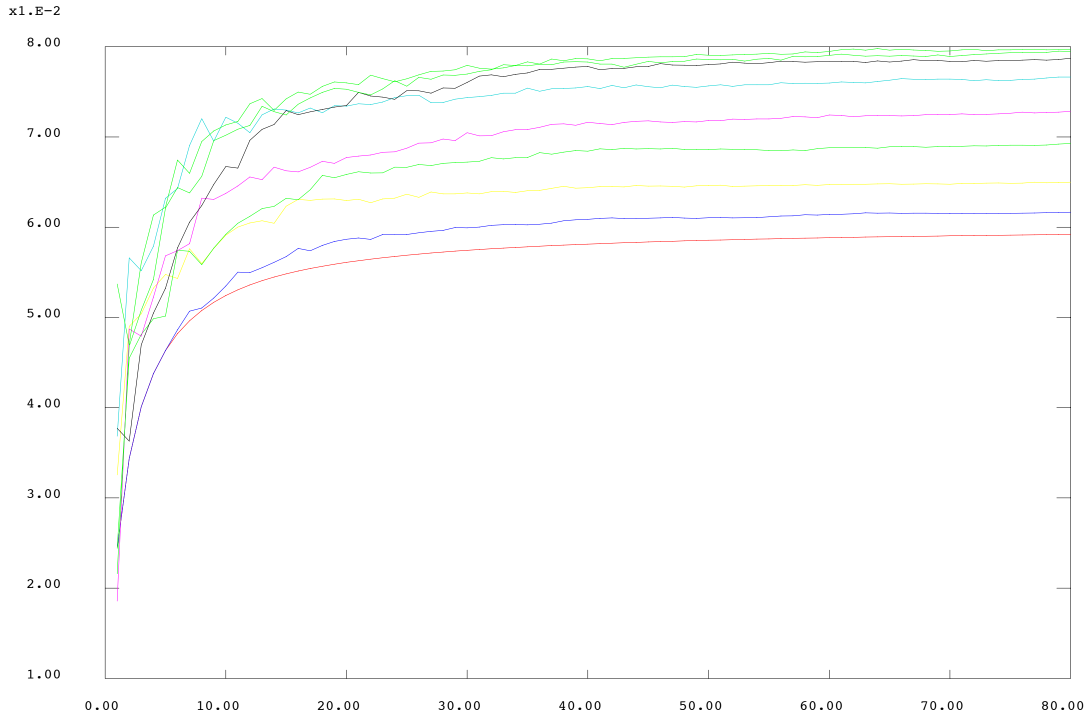

Figure 4: Decreasing of the curves $n \mapsto \Lambda_n(\omega)$ for one realization ω when the probability presence α_0 increases towards 1.

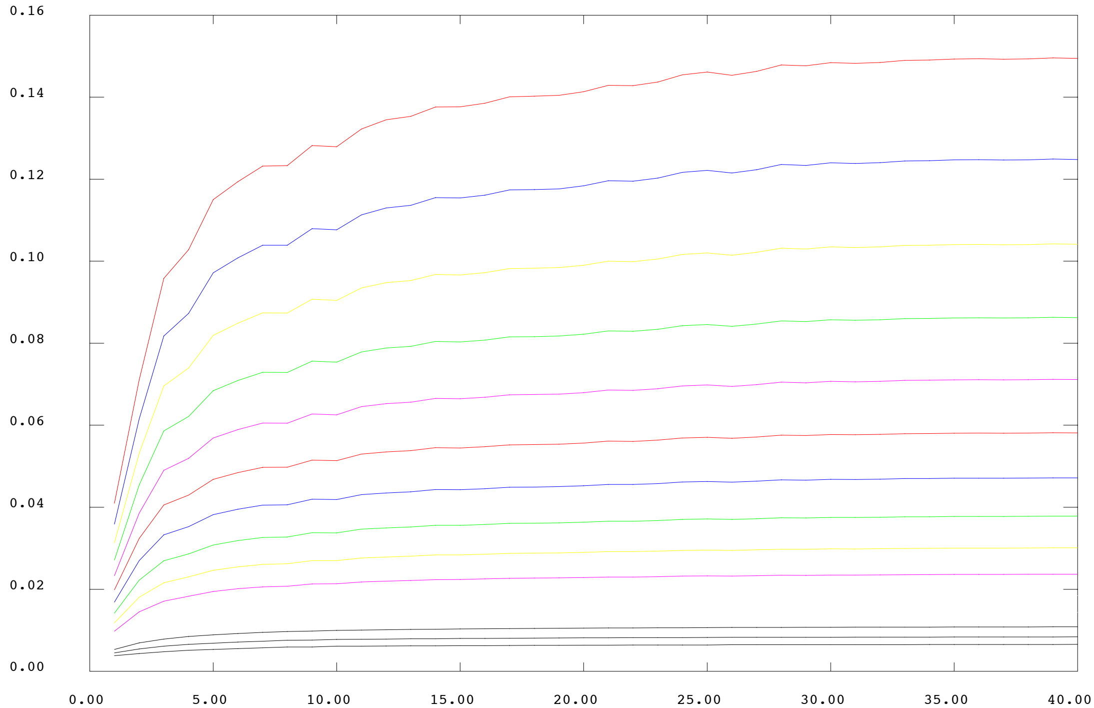

Figure 5: The curves $n \mapsto \Lambda_n(\omega)$ when the radius $\frac{d}{2}$ of the sections increases from 0.2 cm to 0.8 cm with a step 0.04

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