Université de Nîmes

Equipe MIPA
Université de Nîmes, Site des Carmes
Place Gabriel Péri, 30021 Nîmes, France
http://mipa.unimes.fr

Asymptotics of the s-perimeter as $s \searrow 0$

by

Serena Dipierro, Alessio Figalli, Giampiero Palatucci
and Enrico Valdinoci

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SERENA DIPIERRO, ALESSIO FIGALLI, GIAMPIERO PALATUCCI,
AND ENRICO VALDINOCI

ABSTRACT. We deal with the asymptotic behavior of the s-perimeter of a set E inside a domain Ω as $s \searrow 0$. We prove necessary and sufficient conditions for the existence of such limit, by also providing an explicit formulation in terms of the Lebesgue measure of E and Ω. Moreover, we construct examples of sets for which the limit does not exist.

1. INTRODUCTION

Given $s \in (0, 1)$ and a bounded open set $\Omega \subset \mathbb{R}^n$, the s-perimeter of a (measurable) set $E \subseteq \mathbb{R}^n$ in Ω is defined as

$$\begin{aligned} \mathrm{Per}_s(E;\Omega) := {} & L(E\cap\Omega, (\mathscr{C}E)\cap\Omega) \\ & + L(E\cap\Omega, (\mathscr{C}E)\cap(\mathscr{C}\Omega)) + L(E\cap(\mathscr{C}\Omega), (\mathscr{C}E)\cap\Omega), \end{aligned} \tag{1.1}$$

where $\mathscr{C}E = \mathbb{R}^n \setminus E$ denotes the complement of E, and $L(A, B)$ denotes the following nonlocal interaction term

$$L(A,B) := \int_A \int_B \frac{1}{|x-y|^{n+s}}\, dx\, dy \qquad \forall\, A, B \subseteq \mathbb{R}^n. \tag{1.2}$$

This notion of s-perimeter and the related minimization problem was introduced in [4] (see also the pioneering work [14, 15], where some functionals related to the one in (1.1) have been analyzed in connection with fractal dimensions).

Recently, the s-perimeter has inspired a variety of literature in different directions, both in the pure mathematical settings (for instance, as regards the regularity of surfaces with minimal s-perimeter, see [7, 1, 3, 13]) and in view of concrete applications (such as phase transition problems with long range interactions, see [5, 11, 12]). In general, the nonlocal behavior of the functional is the source of major difficulties, conceptual differences, and challenging technical complications. We refer to [9] for an introductory review on this subject.

The limits as $s \searrow 0$ and $s \nearrow 1$ are somehow the critical cases for the s-perimeter, since the functional in (1.1) diverges as it is. Nevertheless, when appropriately rescaled, these limits seem to give meaningful information on the problem. In particular, it was shown in [6, 2] that $(1-s)\mathrm{Per}_s$ approaches the classical perimeter

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functional as $s \nearrow 1$ (up to normalizing multiplicative constants), and this implies that surfaces of minimal s-perimeter inherit the regularity properties of the classical minimal surfaces for s sufficiently close to 1 (see [7]).

As far as we know, the asymptotic as $s \searrow 0$ of $s\mathrm{Per}_s$ was not studied yet (see however [10] for some results in this direction), and this is the question that we would like to address in this paper. That is, we are interested in the quantity

$$\mu(E) := \lim_{s \searrow 0} \mathrm{Per}_s(E;\Omega) \tag{1.3}$$

whenever the limit exists. Of course, if it exists then

$$\mu(E) = \mu(\mathscr{C}E),$$

since

$$\mathrm{Per}_s(E;\Omega) = \mathrm{Per}_s(\mathscr{C}E;\Omega).$$

We will show that, though μ is subadditive (see Proposition 2.1 below), in general it is not a measure (see Proposition 2.2, and this is a major difference with respect to the setting in [10]). On the other hand, μ is additive on bounded, separated sets, and it agrees with the Lebesgue measure of $E \cap \Omega$ (up to normalization) when E is bounded (see Corollary 2.5). As we will show below, a precise characterization of $\mu(E)$ will be given in terms of the behavior of the set E towards infinity, which is encoded in the quantity

$$\alpha(E) := \lim_{s \searrow 0} s \int_{E \cap (\mathscr{C}B_1)} \frac{1}{|y|^{n+s}}\, dy,$$

whenever it exists (see Theorem 2.4 and Corollary 2.5). In fact, the existence of the limit defining α is in general equivalent to the one defining μ (see Proposition 2.6).

As a counterpart of these results, we will construct an explicit example of set E for which both the limits $\mu(E)$ and $\alpha(E)$ do not exist (see Example 2.7): this says that the assumptions we take cannot, in general, be removed.

Also, notice that, in order to make sense of the limit in (1.3), it is necessary to assume that[1]

$$\mathrm{Per}_{s_0}(E;\Omega) < \infty, \text{ for some } s_0 \in (0,1). \tag{1.4}$$

To stress that (1.4) cannot be dropped, we will construct a simple example in which such a condition is violated (see Example 2.8).

The paper is organized as follows. In the following section, we collect the precise statements of all the results we mentioned above. Section 3 is devoted to the proofs.

2. List of the main results

We define $\mathscr{E}$ to be the family of sets $E \subseteq \mathbb{R}^n$ for which the limit defininining $\mu(E)$ in (1.3) exists. We prove the following result:

Proposition 2.1. *μ is subadditive on $\mathscr{E}$, i.e. $\mu(E \cup F) \le \mu(E) + \mu(F)$ for any E, $F \in \mathscr{E}$.*

[1] It is easily seen that if (1.4) holds, then $\mathrm{Per}_s(E;\Omega) < \infty$ for any $s \in (0, s_0)$. Moreover, if ∂E is smooth, then (1.4) is always satisfied.

Now it is convenient to consider the normalized Lebesgue measure $\mathscr{M}$, that is the standard Lebesgue measure scaled by the factor $\mathscr{H}^{n-1}(S^{n-1})$, namely

$$\mathscr{M}(E) := \mathscr{H}^{n-1}(S^{n-1})\,|E|. \tag{2.1}$$

An easy consequence of the results in [10] is that when $E \in \mathscr{E}$ and $E \subseteq \Omega$ then $\mu(E)$ agrees with $\mathscr{M}(E)$ (in fact, we will generalize this statement in Theorem 2.4 and Corollary 2.5). Based on this property valid for subsets of Ω, one may be tempted to infer that μ is always related to the Lebesgue measure, up to normalization, or at least to some more general type of measures. The next result points out that this cannot be true:

Proposition 2.2. *μ is not necessarily additive on separated sets in $\mathscr{E}$, i.e. there exist $E, F \in \mathscr{E}$ such that $\operatorname{dist}(E,F) \geq c > 0$, but $\mu(E \cup F) < \mu(E) + \mu(F)$.*

Also, μ is not necessarily monotone on $\mathscr{E}$, i.e. it is not true that $E \subseteq F$ implies $\mu(E) \leq \mu(F)$.

In particular, we deduce from Proposition 2.2 that μ is not a measure. On the other hand, in some circumstances the additivity property holds true:

Proposition 2.3. *μ is additive on bounded, separated sets in $\mathscr{E}$, i.e. if $E, F \in \mathscr{E}$, E and F are bounded, disjoint and $\operatorname{dist}(E,F) \geq c > 0$, then $E \cup F \in \mathscr{E}$ and $\mu(E \cup F) = \mu(E) + \mu(F)$.*

There is a natural condition under which $\mu(E)$ does exist, based on the weighted volume of E towards infinity, as next result points out:

Theorem 2.4. *Suppose that $\operatorname{Per}_{s_0}(E;\Omega) < \infty$ for some $s_0 \in (0,1)$, and that the following limit exists*

$$\alpha(E) := \lim_{s \searrow 0} s \int_{E \cap (\mathscr{C} B_1)} \frac{1}{|y|^{n+s}}\,dy. \tag{2.2}$$

Then $E \in \mathscr{E}$ and

$$\mu(E) = \big(1 - \tilde{\alpha}(E)\big)\,\mathscr{M}(E \cap \Omega) + \tilde{\alpha}(E)\,\mathscr{M}(\Omega \setminus E),$$

where

$$\tilde{\alpha}(E) := \frac{\alpha(E)}{\mathscr{H}^{n-1}(S^{n-1})}. \tag{2.3}$$

As a consequence of Theorem 2.4, one obtains the existence and the exact expression of $\mu(E)$ for a bounded set E, as described by the following result:

Corollary 2.5. *Let E be a bounded set, and $\operatorname{Per}_{s_0}(E;\Omega) < \infty$ for some $s_0 \in (0,1)$. Then $E \in \mathscr{E}$ and*

$$\mu(E) = \mathscr{M}(E \cap \Omega).$$

In particular, if $E \subseteq \Omega$ and $\operatorname{Per}_{s_0}(E;\Omega) < \infty$ for some $s_0 \in (0,1)$, then $\mu(E) = \mathscr{M}(E)$.

Condition (2.2) is also in general necessary for the existence of the limit in (1.3):

Proposition 2.6. *Suppose that* $\mathrm{Per}_{s_0}(E;\Omega) < \infty$*, for some* $s_0 \in (0,1)$*, and* $|\Omega \setminus E| \neq |E \cap \Omega|$*. Suppose that* $E \in \mathscr{E}$*. Then the limit in* (2.2) *exists and*

$$\alpha(E) = \frac{\mu(E) - \mathscr{M}(E \cap \Omega)}{|\Omega \setminus E| - |E \cap \Omega|}.$$

In the statements above we assumed the existence of the limits in (1.3) and (2.2). Such assumptions cannot be removed, since the limits in (1.3) and (2.2) may not exist, as we now point out:

Example 2.7. *There exists a set* E *for which the limits in* (1.3) *and* (2.2) *do not exist.*

Also, as regards condition (1.4), we point out that it cannot be dropped, since there are sets that do not satisfy it (and for them the limit in (1.3) does not make sense):

Example 2.8. *There exists a set* E *for which* $\mathrm{Per}_s(E;\Omega) = +\infty$ *for any* $s \in (0,1)$*.*

3. Proofs

3.1. **Proof of Proposition 2.1.** We observe that

$$\text{the } s\text{-perimeter is subadditive.} \tag{3.1}$$

To check this, let Ω_1, Ω_2 be open sets of $\mathbb{R}^n$. We remark that

$$\begin{aligned} &L((E\cup F)\cap\Omega_1, (\mathscr{C}(E\cup F))\cap\Omega_2) \\ &\qquad = L((E\cap\Omega_1)\cup(F\cap\Omega_1), (\mathscr{C}E)\cap(\mathscr{C}F)\cap\Omega_2) \\ &\qquad \le L(E\cap\Omega_1, (\mathscr{C}E)\cap(\mathscr{C}F)\cap\Omega_2) + L(F\cap\Omega_1, (\mathscr{C}E)\cap(\mathscr{C}F)\cap\Omega_2) \\ &\qquad \le L(E\cap\Omega_1, (\mathscr{C}E)\cap\Omega_2) + L(F\cap\Omega_1, (\mathscr{C}F)\cap\Omega_2). \end{aligned}$$

By taking $\Omega_1 := \Omega$ and $\Omega_2 := \mathbb{R}^n$ we obtain

$$L((E\cup F)\cap\Omega, \mathscr{C}(E\cup F)) \le L(E\cap\Omega, \mathscr{C}E) + L(F\cap\Omega, \mathscr{C}F),$$

while, by taking $\Omega_1 := \mathscr{C}\Omega$ and $\Omega_2 := \Omega$, we conclude that

$$L((E\cup F)\cap(\mathscr{C}\Omega), (\mathscr{C}(E\cup F))\cap\Omega) \le L(E\cap(\mathscr{C}\Omega), (\mathscr{C}E)\cap\Omega) + L(F\cap(\mathscr{C}\Omega), (\mathscr{C}F)\cap\Omega).$$

By summing up, we get

$$\begin{aligned} &\mathrm{Per}_s(E\cup F;\Omega) \\ &\qquad = L((E\cup F)\cap\Omega, (\mathscr{C}(E\cup F)) + L((E\cup F)\cap(\mathscr{C}\Omega), (\mathscr{C}(E\cup F))\cap\Omega) \\ &\qquad \le L(E\cap\Omega, \mathscr{C}E) + L(F\cap\Omega, \mathscr{C}F) \\ &\qquad\quad + L(E\cap(\mathscr{C}\Omega), (\mathscr{C}E)\cap\Omega) + L(F\cap(\mathscr{C}\Omega), (\mathscr{C}F)\cap\Omega) \\ &\qquad = \mathrm{Per}_s(E;\Omega) + \mathrm{Per}_s(F;\Omega). \end{aligned}$$

This establishes (3.1) and then Proposition 2.1 follows by taking the limit as $s \searrow 0$. □

3.2. **Proof of Proposition 2.2.** First we show that μ is not additive. For this, we observe that if $x \in B_1$ and $y \in \mathscr{C}B_2$ then $|x-y| \le |x|+|y| \le 2|y|$, therefore

$$sL(B_1, \mathscr{C}B_2) \ge c_1 s \int_{B_1} dx \int_{\mathscr{C}B_2} dy \frac{1}{|y|^{n+s}} \ge c_2 s \int_2^{+\infty} \frac{d\rho}{\rho^{1+s}} \ge c_3,$$

for some positive constants c_1, c_2 and c_3. Now we take $E := \mathscr{C}B_2$, $F := \Omega := B_1$. Then

$$\begin{aligned} & \mathrm{Per}_s(E;\Omega) = L(B_1, \mathscr{C}B_2), \\ & \mathrm{Per}_s(F;\Omega) = L(B_1, \mathscr{C}B_1) = L(B_1, \mathscr{C}B_2) + L(B_1, B_2 \setminus B_1) \\ \text{and} \qquad & \mathrm{Per}_s(E \cup F;\Omega) = L(B_1, B_2 \setminus B_1). \end{aligned}$$

Therefore

$$\begin{aligned} s\,\mathrm{Per}_s(E;\Omega) + s\,\mathrm{Per}_s(F;\Omega) &= 2sL(B_1, \mathscr{C}B_2) + sL(B_1, B_2 \setminus B_1) \\ &\ge 2c_3 + s\,L(B_1, B_2 \setminus B_1) \\ &= 2c_3 + s\,\mathrm{Per}_s(E \cup F;\Omega). \end{aligned}$$

By sending $s \searrow 0$, we conclude that $\mu(E) + \mu(F) \ge 2c_3 + \mu(E \cup F)$, so μ is not additive.

Now we show that μ is not monotone either. For this we take E such that $\mu(E) > 0$ (for instance, one can take E a small ball inside Ω; see Corollary 2.5), and $F := \mathbb{R}^n$: with this choice, $E \subset F$ and $\mathrm{Per}_s(F;\Omega) = 0$, so $\mu(E) > 0 = \mu(F)$. □

3.3. **Auxiliary observations.** Here we collect some observations, to be exploited in the subsequent proofs.

Observation 1. First of all, we observe that

$$\text{if } A \text{ and } B \text{ are bounded, disjoints sets with } \mathrm{dist}(A,B) \ge c > 0, \text{ then } \lim_{s \searrow 0} s\,L(A,B) = 0. \tag{3.2}$$

To check this, suppose that A and B lie in B_R. Then

$$\int_A \int_B \frac{1}{|x-y|^{n+s}}\,dx\,dy \le \int_{B_R} \int_{B_R} \frac{1}{c^{n+s}}\,dx\,dy \le \frac{(\mathscr{H}^{n-1}(S^{n-1}))^2 R^{2n}}{c^{n+s}}$$

and this establishes (3.2).

Observation 2. Now we would like to remark that the quantity

$$\lim_{s \searrow 0} s \int_{E \cap (\mathscr{C}B_R)} \frac{1}{|y|^{n+s}}\,dy$$

is independent of R, if the limit exists. More precisely, we show that for any $R \ge r > 0$

$$\lim_{s \searrow 0} s \left(\int_{E \cap (\mathscr{C}B_R)} \frac{1}{|y|^{n+s}}\,dy - \int_{E \cap (\mathscr{C}B_r)} \frac{1}{|y|^{n+s}}\,dy \right) = 0. \tag{3.3}$$

To prove this, we notice that

$$\begin{aligned} s\int_{E\cap(B_R\setminus B_r)}\frac{1}{|y|^{n+s}}\,dy \le s\int_{B_R\setminus B_r}\frac{1}{|y|^{n+s}}\,dy &= s\mathscr{H}^{n-1}(S^{n-1})\int_r^R\frac{1}{\rho^{1+s}}\,d\rho \\ &= \mathscr{H}^{n-1}(S^{n-1})\left(\frac{1}{r^s}-\frac{1}{R^s}\right) \end{aligned} \tag{3.4}$$

and so, by taking limit in s,

$$\lim_{s\searrow 0} s\int_{E\cap(B_R\setminus B_r)}\frac{1}{|y|^{n+s}}\,dy = 0,$$

which establishes (3.3).

Observation 3. As a consequence of (3.3), it follows that if the limit in (2.2) exists then

$$\alpha(E) = \lim_{s\searrow 0} s\int_{E\cap(\mathscr{C}B_R)}\frac{1}{|y|^{n+s}}\,dy \qquad \forall\, R>0. \tag{3.5}$$

Observation 4. For any $s\in(0,1)$, we define

$$\alpha_s(E) := s\int_{E\cap(\mathscr{C}B_1)}\frac{1}{|y|^{n+s}}\,dy \tag{3.6}$$

and we prove that, for any bounded set $F\subset\mathbb{R}^n$, and any set $E\subseteq\mathbb{R}^n$,

$$\lim_{R\to+\infty}\limsup_{s\searrow 0}\left|\alpha_s(E)\,|F| - s\int_F\int_{E\cap(\mathscr{C}B_R)}\frac{1}{|x-y|^{n+s}}\,dx\,dy\right| = 0. \tag{3.7}$$

To prove this, we take $r>0$ such that $F\subset B_r$ and $R>1+2r$ (later on R will be taken as large as we wish). We observe that, for any $z\in B_r$ and $y\in\mathscr{C}B_R$,

$$|z-y|\ge|y|-|z| = \left(1-\frac{r}{R}\right)|y| + \frac{r}{R}|y| - |z| \ge \frac{|y|}{2}.$$

Therefore, if, for any fixed $y\in\mathscr{C}B_R$ we consider the map

$$h(z) := \frac{1}{|z-y|^{n+s}}, \qquad z\in B_r,$$

we have that

$$|\nabla h(z)| = \frac{n+s}{|z-y|^{n+s+1}} \le \frac{2^{n+s+1}(n+s)}{|y|^{n+s+1}},$$

for any $z\in B_r$, which implies

$$\left|\frac{1}{|x-y|^{n+s}} - \frac{1}{|y|^{n+s}}\right| = |h(x)-h(0)| \le \frac{2^{n+s+1}(n+s)|x|}{|y|^{n+s+1}} \qquad \forall\, x\in B_r,\ y\in\mathscr{C}B_R.$$

Therefore

$$\begin{aligned}
&\left|\int_F \left(\int_{E\cap(\mathscr{C}B_R)} \frac{1}{|y|^{n+s}}\,dy\right)dx - \int_F\int_{E\cap(\mathscr{C}B_R)} \frac{1}{|x-y|^{n+s}}\,dx\,dy\right| \\
&\quad\le \int_F \left(\int_{E\cap(\mathscr{C}B_R)} \left|\frac{1}{|y|^{n+s}} - \frac{1}{|x-y|^{n+s}}\right| dy\right)dx \\
&\quad\le \int_F \left(\int_{E\cap(\mathscr{C}B_R)} \frac{2^{n+s+1}(n+s)|x|}{|y|^{n+s+1}}\,dy\right)dx \\
&\quad\le 2^{n+s+1}(n+s)|F|r\int_{\mathscr{C}B_R} \frac{1}{|y|^{n+s+1}}\,dy \;\le\; C
\end{aligned}$$

for some $C > 0$ independent of s. As a consequence

$$\begin{aligned}
&\left|\alpha_s(E)\,|F| - s\int_F\int_{E\cap(\mathscr{C}B_R)} \frac{1}{|x-y|^{n+s}}\,dx\,dy\right| \\
&\qquad\qquad \le |F|\left|\alpha_s(E) - s\int_{E\cap(\mathscr{C}B_R)} \frac{1}{|y|^{n+s}}\,dy\right| + Cs.
\end{aligned}$$

This and (3.3) (applied here with $r := 1$) imply (3.7).

Observation 5. If the limit in (2.2) exists, then (3.7) boils down to

$$\lim_{R\to+\infty}\limsup_{s\searrow 0}\left|\alpha(E)\,|F| - s\int_F\int_{E\cap(\mathscr{C}B_R)} \frac{1}{|x-y|^{n+s}}\,dx\,dy\right| = 0. \tag{3.8}$$

Observation 6. Now we point out that, $F \subseteq \Omega \subset B_R$ for some $R > 0$, and F has finite s_0-perimeter in Ω for some $s_0 \in (0,1)$, then

$$\lim_{s\searrow 0} s\int_F\int_{B_R\setminus F} \frac{1}{|x-y|^{n+s}}\,dx\,dy = 0. \tag{3.9}$$

Indeed, for any $s \in (0, s_0)$,

$$\begin{aligned}
&\int_F\int_{B_R\setminus F} \frac{1}{|x-y|^{n+s}}\,dx\,dy \\
&\quad\le \int_F\int_{(B_R\setminus F)\cap\{|x-y|\le 1\}} \frac{1}{|x-y|^{n+s_0}}\,dx\,dy + \int_F\int_{(B_R\setminus F)\cap\{|x-y|>1\}} 1\,dx\,dy \\
&\quad\le \mathrm{Per}_{s_0}(F;\Omega) + |B_R|^2,
\end{aligned}$$

which implies (3.9).

Observation 7. Let $E_1 := E \cap \Omega$ and $E_2 := E \setminus E_1$. Then

$$\begin{aligned}
\mathrm{Per}_s(E;\Omega) &= \mathrm{Per}_s(E_1 \cup E_2;\Omega) \\
&= L(E_1, \Omega\setminus E_1) + L(E_1, (\mathscr{C}\Omega)\setminus E_2) + L(E_2, \Omega\setminus E_1) \\
&= L(E_1, \mathscr{C}E_1) - L(E_1, E_2) + L(E_2, \Omega\setminus E_1) \\
&= \mathrm{Per}_s(E_1;\Omega) - L(E_1, E_2) + L(E_2, \Omega\setminus E_1).
\end{aligned} \tag{3.10}$$

With these observations in hand, we are ready to continue the proofs of the main results.

3.4. **Proof of Proposition 2.3.** We prove Proposition 2.3 by suitably modifying the proof of Proposition 2.1. Given two open sets Ω_1 and Ω_2, and two disjoint sets E and F, we have that

$$\begin{aligned} & L((E\cup F)\cap\Omega_1,(\mathscr{C}(E\cup F))\cap\Omega_2) \\ &\qquad = L((E\cap\Omega_1)\cup(F\cap\Omega_1),(\mathscr{C}E)\cap(\mathscr{C}F)\cap\Omega_2) \\ &\qquad = L(E\cap\Omega_1,(\mathscr{C}E)\cap(\mathscr{C}F)\cap\Omega_2)+L(F\cap\Omega_1,(\mathscr{C}E)\cap(\mathscr{C}F)\cap\Omega_2). \end{aligned}$$

By taking $\Omega_1 := \Omega$ and $\Omega_2 := \mathbb{R}^n$ we obtain

$$L((E\cup F)\cap\Omega,\mathscr{C}(E\cup F)) = L(E\cap\Omega,(\mathscr{C}E)\cap(\mathscr{C}F)) + L(F\cap\Omega,(\mathscr{C}E)\cap(\mathscr{C}F))$$

while, by taking $\Omega_1 := \mathscr{C}\Omega$ and $\Omega_2 := \Omega$, we conclude that

$$\begin{aligned} & L((E\cup F)\cap(\mathscr{C}\Omega),(\mathscr{C}(E\cup F))\cap\Omega) \\ &\qquad = L(E\cap(\mathscr{C}\Omega),(\mathscr{C}E)\cap(\mathscr{C}F)\cap\Omega)+L(F\cap(\mathscr{C}\Omega),(\mathscr{C}E)\cap(\mathscr{C}F)\cap\Omega). \end{aligned}$$

As a consequence,

$$\begin{aligned} & \mathrm{Per}_s(E\cup F;\Omega) \\ &\quad = L((E\cup F)\cap\Omega,\mathscr{C}(E\cup F))+L((E\cup F)\cap(\mathscr{C}\Omega),(\mathscr{C}(E\cup F))\cap\Omega) \\ &\quad = L(E\cap\Omega,(\mathscr{C}E)\cap(\mathscr{C}F))+L(F\cap\Omega,(\mathscr{C}E)\cap(\mathscr{C}F)) \\ &\qquad +L(E\cap(\mathscr{C}\Omega),(\mathscr{C}E)\cap(\mathscr{C}F)\cap\Omega)+L(F\cap(\mathscr{C}\Omega),(\mathscr{C}E)\cap(\mathscr{C}F)\cap\Omega) \\ &\quad = \mathrm{Per}_s(E;\Omega)+\mathrm{Per}_s(F;\Omega) \\ &\qquad -L(E\cap\Omega,(\mathscr{C}E)\cap F)-L(F\cap\Omega,E\cap(\mathscr{C}F)) \\ &\qquad -L(E\cap(\mathscr{C}\Omega),(\mathscr{C}E)\cap F\cap\Omega)-L(F\cap(\mathscr{C}\Omega),E\cap(\mathscr{C}F)\cap\Omega). \end{aligned}$$

We remark that the last interactions involve only bounded, separated sets, since so are E and F, therefore, by (3.2),

$$\lim_{s\searrow 0} s\,\mathrm{Per}_s(E\cup F;\Omega) = \lim_{s\searrow 0}\big(s\,\mathrm{Per}_s(E;\Omega)+s\,\mathrm{Per}_s(F;\Omega)\big)+0,$$

which completes the proof of Proposition 2.3. □

3.5. **Proof of Theorem 2.4.** We suppose that $\Omega\subset B_r$, for some $r>0$, and we take $R>1+2r$. Let $E_1 := E\cap\Omega$ and $E_2 := E\setminus E_1$. Notice that, for any $F\subseteq\Omega$,

$$E_2\cap B_R\subseteq B_R\setminus\Omega\subseteq B_R\setminus F$$

and so (3.9) gives that

$$\lim_{s\searrow 0} s\int_F\int_{E_2\cap B_R}\frac{1}{|x-y|^{n+s}}\,dx\,dy = 0.$$

Using this and (3.8), we conclude that, for any $F \subseteq \Omega$,

$$\begin{aligned}\lim_{s\searrow 0} s \int_F \int_{E_2} \frac{1}{|x-y|^{n+s}}\,dx\,dy &= \lim_{R\to+\infty} \lim_{s\searrow 0} s \int_F \int_{E_2} \frac{1}{|x-y|^{n+s}}\,dx\,dy \\ &= \lim_{R\to+\infty} \lim_{s\searrow 0} s \int_F \int_{E_2\cap(\mathscr{C}B_R)} \frac{1}{|x-y|^{n+s}}\,dx\,dy \\ &= \alpha(E)\,|F|.\end{aligned}$$

In particular, by taking $F := E_1$ and $F := \Omega \setminus E_1$, and recalling (2.1) and (2.3),

$$\begin{aligned}&\lim_{s\searrow 0} s \int_{E_1} \int_{E_2} \frac{1}{|x-y|^{n+s}}\,dx\,dy = \alpha(E)\,|E_1| = \tilde\alpha(E)\,\mathscr{M}(E_1) \\ \text{and}\quad &\lim_{s\searrow 0} s \int_{\Omega\setminus E_1} \int_{E_2} \frac{1}{|x-y|^{n+s}}\,dx\,dy = \alpha(E)\,|\Omega\setminus E_1| = \tilde\alpha(E)\,\mathscr{M}(\Omega\setminus E_1).\end{aligned} \tag{3.11}$$

We now claim

$$\lim_{s\searrow 0} s\,\mathrm{Per}_s(E_1;\Omega) = \mathscr{M}(E_1). \tag{3.12}$$

Indeed, since $E_1 \subseteq \Omega$, this is a plain consequence of Theorem 3 in [10] (see also Remark 4.3 in [8] for another elementary proof) by simply choosing $u = \chi_{E_1}$ there:

$$\begin{aligned}\lim_{s\searrow 0} s\,\mathrm{Per}_s(E_1;\Omega) &= \lim_{s\searrow 0} s\,L(E_1,\mathscr{C}E_1) \\ &= \lim_{s\searrow 0} \frac{s}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\chi_{E_1}(x)-\chi_{E_1}(y)|^2}{|x-y|^{n+s}}\,dx\,dy \\ &= \mathscr{H}^{n-1}(S^{n-1})\,\|\chi_{E_1}\|^2_{L^2(\mathbb{R}^n)} \;=\; \mathscr{H}^{n-1}(S^{n-1})\,|E_1|,\end{aligned}$$

as desired. Thus, using (3.10), (3.11), and (3.12), we obtain

$$\lim_{s\searrow 0} s\mathrm{Per}_s(E;\Omega) = \mathscr{M}(E_1) - \tilde\alpha(E)\mathscr{M}(E_1) + \tilde\alpha(E)\mathscr{M}(\Omega\setminus E_1),$$

which is the desired result. □

3.6. **Proof of Corollary 2.5.** We fix R large enough so that $E \subset B_R$, so that $E \cap (\mathscr{C}B_R) = \emptyset$. By the expression of $\alpha(E)$ in (3.5), we have that the limit in (2.2) exists and $\alpha(E) = 0$. Then the result follows by Theorem 2.4. □

3.7. **Proof of Proposition 2.6.** We suppose that $\Omega \subset B_r$, for some $r > 0$, and we take $R > 1 + 2r$. Let $E_1 := E \cap \Omega$ and $E_2 := E \setminus E_1$. By (3.10),

$$\begin{aligned}&s\mathrm{Per}_s(E;\Omega) - s\mathrm{Per}_s(E_1;\Omega) \\ &\quad = sL(E_2,\Omega\setminus E_1) - sL(E_1,E_2) \\ &\quad = s\int_{\Omega\setminus E_1}\int_{E_2\cap B_R} \frac{1}{|x-y|^{n+s}}\,dx\,dy + s\int_{\Omega\setminus E_1}\int_{E_2\cap(\mathscr{C}B_R)} \frac{1}{|x-y|^{n+s}}\,dx\,dy \\ &\qquad - s\int_{E_1}\int_{E_2\cap B_R} \frac{1}{|x-y|^{n+s}}\,dx\,dy - s\int_{E_1}\int_{E_2\cap(\mathscr{C}B_R)} \frac{1}{|x-y|^{n+s}}\,dx\,dy.\end{aligned}$$

By rearranging the terms, we obtain

(3.13)
$$\begin{aligned} I(s,R) &:= s\int_{\Omega\setminus E_1}\int_{E_2\cap(\mathscr{C}B_R)}\frac{1}{|x-y|^{n+s}}\,dx\,dy - s\int_{E_1}\int_{E_2\cap(\mathscr{C}B_R)}\frac{1}{|x-y|^{n+s}}\,dx\,dy \\ &= s\mathrm{Per}_s(E;\Omega) - s\mathrm{Per}_s(E_1;\Omega) - s\int_{\Omega\setminus E_1}\int_{E_2\cap B_R}\frac{1}{|x-y|^{n+s}}\,dx\,dy \\ &\quad + s\int_{E_1}\int_{E_2\cap B_R}\frac{1}{|x-y|^{n+s}}\,dx\,dy. \end{aligned}$$

By using (3.9) with $F := \Omega\setminus E_1$ and $F := E_1$, we have that the last two terms in (3.13) converge to zero as $s\searrow 0$, therefore, recalling also Corollary 2.5, we obtain from (3.13) that

$$\lim_{s\searrow 0} I(s,R) = \mu(E) - \mu(E_1). \tag{3.14}$$

We recall the notation in (3.6) and we write

$$\begin{aligned} \alpha_s(E)\,|\Omega\setminus E_1| \;&=\; s\int_{\Omega\setminus E_1}\int_{E_2\cap(\mathscr{C}B_R)}\frac{1}{|x-y|^{n+s}}\,dx\,dy \\ &\quad + \alpha_s(E)\,|\Omega\setminus E_1| - s\int_{\Omega\setminus E_1}\int_{E_2\cap(\mathscr{C}B_R)}\frac{1}{|x-y|^{n+s}}\,dx\,dy, \end{aligned}$$

and

$$\begin{aligned} \alpha_s(E)\,|E_1| \;&=\; s\int_{E_1}\int_{E_2\cap(\mathscr{C}B_R)}\frac{1}{|x-y|^{n+s}}\,dx\,dy \\ &\quad + \alpha_s(E)\,|E_1| - s\int_{E_1}\int_{E_2\cap(\mathscr{C}B_R)}\frac{1}{|x-y|^{n+s}}\,dx\,dy. \end{aligned}$$

By subtracting term by term, we obtain that

$$\begin{aligned} &\alpha_s(E)\Big(|\Omega\setminus E_1| - |E_1|\Big) \\ &\quad = I(s,R) + \left(\alpha_s(E)\,|\Omega\setminus E_1| - s\int_{\Omega\setminus E_1}\int_{E_2\cap(\mathscr{C}B_R)}\frac{1}{|x-y|^{n+s}}\,dx\,dy\right) \\ &\qquad - \left(\alpha_s(E)\,|E_1| - s\int_{E_1}\int_{E_2\cap(\mathscr{C}B_R)}\frac{1}{|x-y|^{n+s}}\,dx\,dy\right). \end{aligned}$$

As a consequence, by using (3.7) (applied here with $F := \Omega\setminus E_1$ and $F := E_1$),

$$\lim_{R\to+\infty}\lim_{s\searrow 0}\Big[\alpha_s(E)\Big(|\Omega\setminus E_1| - |E_1|\Big) - I(s,R)\Big] = 0.$$

Accordingly, by (3.14), we obtain the existence of the limit

$$\begin{aligned}\Big(|\Omega\setminus E_1|-|E_1|\Big)&\lim_{s\searrow 0}\alpha_s(E)\\&=\lim_{R\to+\infty}\lim_{s\searrow 0}\alpha_s(E)\Big(|\Omega\setminus E_1|-|E_1|\Big)\\&=\lim_{R\to+\infty}\lim_{s\searrow 0}\left\{\Big[\alpha_s(E)\Big(|\Omega\setminus E_1|-|E_1|\Big)-I(s,R)\Big]+I(s,R)\right\}\\&=0+\mu(E)-\mu(E_1),\end{aligned}$$

which completes the proof of Proposition 2.6, by recalling that $\mu(E_1)=\mathscr{M}(E\cap\Omega)$, in view of Corollary 2.5. □

3.8. **Construction of Example 2.7.** We start with some preliminary computations. Let $a_k := 10^{k^2}$, for any $k\in\mathbb{N}$. For any $j\in\{0,1,2,3\}$, let

$$I_j := \Big\{x\in\mathbb{R} \text{ s.t. } \exists k\in\mathbb{N} \text{ s.t. } x\in\big[a_{4k+j},a_{4k+j+1}\big)\Big\}.$$

Notice that $[1,+\infty)$ may be written as the disjoint union of the I_j's. Let $\varphi\in C^\infty\big([0,+\infty),[0,1]\big)$ be such that $\varphi=0$ in $[0,1]\cup I_0$, $\varphi=1$ in I_2, and then φ smoothly interpolates between 0 and 1 in $I_1\cup I_3$.

We claim that there exist two sequences $\nu_{0,k}\to+\infty$ and $\nu_{1,k}\to+\infty$ such that

$$\lim_{k\to+\infty}\int_0^{+\infty}\varphi(\nu_{0,k}x)e^{-x}\,dx=0 \quad\text{and}\quad \lim_{k\to+\infty}\int_0^{+\infty}\varphi(\nu_{1,k}x)e^{-x}\,dx=1. \tag{3.15}$$

To check (3.15), we take $\nu_{0,k}:=a_{4k+1}/k$ and $\nu_{1,k}:=a_{4k+3}/k$. We observe that, by construction, $\varphi=0$ in $\big[a_{4k},a_{4k+1}\big)$ and $\varphi=1$ in $\big[a_{4k+2},a_{4k+3}\big)$, so $\varphi(\nu_{0,k}x)=0$ for any $x\in[kb_{0,k},\,k)$ and $\varphi(\nu_{1,k}x)=1$ in $[kb_{1,k},\,k)$, where

$$b_{0,k}:=\frac{a_{4k}}{a_{4k+1}}=10^{-(8k+1)} \quad\text{and}\quad b_{1,k}:=\frac{a_{4k+2}}{a_{4k+3}}=10^{-(8k+5)}.$$

We deduce that

$$\int_0^{+\infty}\varphi(\nu_{0,k}x)e^{-x}\,dx\le\int_0^{kb_{0,k}}e^{-x}\,dx+\int_k^{+\infty}e^{-x}\,dx=1-e^{-kb_{0,k}}+e^{-k}$$

and

$$\int_0^{+\infty}\varphi(\nu_{1,k}x)e^{-x}\,dx\ge\int_{kb_{1,k}}^{k}e^{-x}\,dx=e^{-kb_{1,k}}-e^{-k}.$$

This implies (3.15) by noticing that

$$\lim_{k\to+\infty}kb_{0,k}=0=\lim_{k\to+\infty}kb_{1,k}.$$

Now we construct our example by using the above function φ and (3.15). We take $\Omega:=B_{1/2}$ and $E:=\big\{x=(\rho\cos\gamma,\rho\sin\gamma),\rho>1,\gamma\in[0,\theta(\rho)]\big\}\subset\mathbb{R}^2$, where $\theta(\rho):=\varphi(\log\rho)$. Then, recalling (3.6) we have

$$\alpha_s(E)=s\int_1^{+\infty}\int_0^{\theta(\rho)}\frac{\rho^{n-1}}{\rho^{n+s}}\,d\theta\,d\rho=s\int_1^{+\infty}\theta(\rho)\frac{1}{\rho^{1+s}}\,d\rho.$$

Therefore, by the change of variable $\log\rho=r$, we have

$$\alpha_s(E)=s\int_0^{+\infty}\varphi(r)\,e^{-rs}\,dr,$$

and, by the further change $rs = x$, we have

$$\alpha_s(E) = \int_0^{+\infty} \varphi\left(\frac{x}{s}\right) e^{-x}\, dx.$$

If we set $\nu = 1/s$, the limit in (2.2) becomes the following:

$$\alpha(E) = \lim_{\nu\to\infty} \int_0^{+\infty} \varphi(\nu x)\, e^{-x}\, dx,$$

and, by (3.15), we get that such a limit does not exist. This shows that the limit in (2.2) does not exist. Since $|\Omega \setminus E| = |B_{1/2}| > 0 = |E \cap \Omega|$, by Proposition 2.6, the limit in (1.3) does not exist either. $\square$

3.9. **Construction of Example 2.8.** We take a decreasing sequence β_k such that $\beta_k > 0$ for any $k \geq 1$,

$$L := \sum_{k=1}^{+\infty} \beta_k < +\infty$$

but

$$\sum_{k=1}^{+\infty} \beta_{2k}^{1-s} = +\infty \qquad \forall\, s \in (0,1). \tag{3.16}$$

For instance, one can take $\beta_1 := \dfrac{1}{\log^2 2}$ and $\beta_k := \dfrac{1}{k \log^2 k}$ for any $k \geq 2$.

Now, we define

$$\begin{aligned} &\Omega := (0, L) \subset \mathbb{R},\\ &\sigma_m := \sum_{k=1}^{m} \beta_k,\\ &I_m := (\sigma_m, \sigma_{m+1}),\\ \text{and} \qquad &E := \bigcup_{j=1}^{+\infty} I_{2j}. \end{aligned}$$

Notice that $E \subset \Omega$ and

$$\begin{aligned} \mathrm{Per}_s(E;\Omega) &= L(E, \mathscr{C}E)\\ &\geq \sum_{j=1}^{+\infty} L(I_{2j}, I_{2j+1}) = \sum_{j=1}^{+\infty} \int_{\sigma_{2j}}^{\sigma_{2j+1}} \int_{\sigma_{2j+1}}^{\sigma_{2j+2}} \frac{1}{|x-y|^{1+s}}\, dx\, dy. \end{aligned} \tag{3.17}$$

An integral computation shows that if $a < b < c$ then

$$\int_a^b \int_b^c \frac{1}{|x-y|^{1+s}}\, dx\, dy = \frac{1}{s(1-s)} \Big[(c-b)^{1-s} + (b-a)^{1-s} - (c-a)^{1-s} \Big].$$

By plugging this into (3.17), we obtain

$$\begin{aligned} & s(1-s)\mathrm{Per}_s(E;\Omega) \\ & \quad \geq \sum_{j=1}^{+\infty} \Big[(\sigma_{2j+2}-\sigma_{2j+1})^{1-s} + (\sigma_{2j+1}-\sigma_{2j})^{1-s} - (\sigma_{2j+2}-\sigma_{2j})^{1-s} \Big] \\ & \quad = \sum_{j=1}^{+\infty} \beta_{2j+2}^{1-s} + \beta_{2j+1}^{1-s} - (\beta_{2j+2}+\beta_{2j+1})^{1-s}. \end{aligned} \tag{3.18}$$

Now we observe that the map $[0,1) \ni t \mapsto (1+t)^{1-s}$ is concave, therefore

$$(1+t)^{1-s} \leq 1 + (1-s)t \leq 1 + (1-s)t^{1-s}$$

for any $t \in [0,1)$, that is

$$1 + t^{1-s} - (1+t)^{1-s} \geq s t^{1-s}.$$

By taking $t := \beta_{2j+2}/\beta_{2j+1}$ and then multiplying by β_{2j+1}^{1-s}, we obtain

$$\beta_{2j+1}^{1-s} + \beta_{2j+2}^{1-s} - (\beta_{2j+1}+\beta_{2j+2})^{1-s} \geq s\beta_{2j+2}^{1-s}.$$

By plugging this into (3.18) and using (3.16), we conclude that

$$\mathrm{Per}_s(E;\Omega) \geq \frac{1}{1-s}\sum_{j=1}^{+\infty} \beta_{2j+2}^{1-s} = +\infty \qquad \forall\, s \in (0,1),$$

as desired. □

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(Serena Dipierro) SISSA, Sector of Mathematical Analysis, Via Bonomea, 265, 34136 Trieste, Italy

E-mail address: dipierro@sissa.it

(Alessio Figalli) University of Texas at Austin, Department of Mathematics, 2515 Speedway Stop C1200, Austin, TX 78712-1202, USA

E-mail address: figalli@math.utexas.edu

(Giampiero Palatucci) Dipartimento di Matematica, Università degli Studi di Parma, Campus - Parco Area delle Scienze, 53/A, 43124 Parma, Italy

E-mail address: giampiero.palatucci@unimes.fr

(Enrico Valdinoci) Dipartimento di Matematica, Università degli Studi di Milano, Via Saldini, 50, 20133 Milano, Italy

E-mail address: enrico@math.utexas.edu