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Lower semicontinuity via $W^{1,q}$-quasiconvexity

by

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LOWER SEMICONTINUITY VIA $W^{1,q}$-QUASICONVEXITY

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Abstract. We isolate a general condition on $L : \mathbb{M} \to [0, \infty]$, assumed to be continuous, under which $W^{1,q}$-quasiconvexity with $q \in [1, \infty]$ is a sufficient condition for $I(u) = \int_\Omega L(\nabla u(x))dx$ to be sequentially weakly lower semicontinuous on $W^{1,p}(\Omega; \mathbb{R}^m)$ with $p \in]1, \infty[$.

1. Introduction

Let $m, N \geq 1$ be two integers, let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary, let $\mathbb{M} := \mathbb{M}^{m\times N}$, where $\mathbb{M}^{m\times N}$ denotes the space of all real $m \times N$ matrices. Let $p \in]1, \infty[$, let $L : \mathbb{M} \to [0, \infty]$ be a continuous function and let $I : W^{1,p}(\Omega; \mathbb{R}^m) \to [0, \infty]$ be defined by

$$I(u) := \int_\Omega L(\nabla u(x))dx.$$

In [BM84] Ball and Murat introduced the concept of $W^{1,q}$-quasiconvexity for $q \in [1, \infty]$, i.e., L is $W^{1,q}$-quasiconvex if and only if

$$\int_Y L\left(\nabla u(y)\right) dy \geq L(\xi) \text{ for all } u \in l_\xi + W^{1,q}_0(Y; \mathbb{R}^m)$$

with $l_\xi(y) := \xi y$ and $Y :=] - \frac{1}{2}, \frac{1}{2}[^N$, and proved (see [BM84, Corollary 3.2]) that $W^{1,p}$-quasiconvexity is a necessary condition for I to be sequentially weakly lower semicontinuous (swlsc) on $W^{1,p}(\Omega; \mathbb{R}^m)$, i.e., when

$$u_n \rightharpoonup u \text{ in } W^{1,p}(\Omega; \mathbb{R}^m) \text{ implies } \varliminf_{n\to\infty} I(u_n) \geq I(u).$$

However, proving that $W^{1,p}$-quasiconvexity, or some variant of it, is also sufficient is still an open problem. In this paper we isolate a general condition on L (see $(\mathrm{C}_{p,q})$ in Theorem 1.1) under which $W^{1,q}$-quasiconvexity is a sufficient condition for I to be swlsc on $W^{1,p}(\Omega; \mathbb{R}^m)$. More precisely, our main result is the following.

Theorem 1.1. *Given $p \in]1, \infty[$ and $q \in [1, \infty]$, assume that L is $W^{1,q}$-quasiconvex and satisfies*

$(\mathrm{C}_{p,q})$ *for every $\xi \in \mathbb{M}$ and every $\{v_n\}_n \subset W^{1,p}(Y; \mathbb{R}^m)$ such that*

$$\begin{cases} v_n \rightharpoonup l_\xi \text{ in } W^{1,p}(Y; \mathbb{R}^m); \\ \sup_n \int_Y L(\nabla v_n(y))dy < \infty, \end{cases}$$

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there exist a subsequence $\{v_n\}_n$ (not relabeled) and $\{w_n\}_n \subset l_\xi + W_0^{1,q}(Y;\mathbb{R}^m)$ such that

$$\begin{cases} |\nabla v_n - \nabla w_n| \to 0 \text{ in measure;} \\ \{L(\nabla w_n)\}_n \text{ is equi-integrable.} \end{cases}$$

Then, I is swlsc on $W^{1,p}(\Omega;\mathbb{R}^m)$.

As a direct consequence of Theorem 1.1, we have

Corollary 1.2. *Given $p \in]1,\infty[$, if $(\mathrm{C}_{p,p})$ holds then $W^{1,p}$-quasiconvexity is a necessary and sufficient condition for I to be swlsc on $W^{1,p}(\Omega;\mathbb{R}^m)$.*

In fact, Acerbi and Fusco (see [AF84]) showed that $W^{1,\infty}$-quasiconvexity is sufficient for I to be swlsc on $W^{1,p}(\Omega;\mathbb{R}^m)$ provided that L has p-growth, i.e., $L(\cdot) \le \alpha(1 + |\cdot|^p)$ for some $\alpha > 0$. We remark that the key argument in their proof is in fact the following result, which we call "localization principle":

(A) for every $\xi \in \mathbb{M}$ and every $\{v_n\}_n \subset W^{1,p}(Y;\mathbb{R}^m)$ such that

$$v_n \rightharpoonup l_\xi \text{ in } W^{1,p}(Y;\mathbb{R}^m),$$

there exist a subsequence $\{v_n\}_n$ (not relabeled) and $\{w_n\}_n \subset l_\xi + C_c^\infty(Y;\mathbb{R}^m)$ such that

$$\begin{cases} |\nabla v_n - \nabla w_n| \to 0 \text{ in measure} \\ \{|\nabla w_n|^p\}_n \text{ is equi-integrable.} \end{cases}$$

Note that (A) is a particular case of the decomposition lemma (for more details see Kristensen [Kri94] and also Fonseca, Müller and Pedregal [FMP98]). Using this "localization principle" Kinderlehrer and Pedregal (see [KP92] and also [Syc99]) proved Acerbi-Fusco's theorem by using Young measure theory. Kinderlehrer-Pedregal's approach was extended by Sychev (see [Syc05]) to the case where L has fast growth, i.e., $\beta G(|\cdot|) \le L(\cdot) \le \alpha(1 + G(|\cdot|))$ for some $\alpha, \beta > 0$ and some convex function $G : [0,\infty[\to [0,\infty[$ such that $\lim_{t\to\infty} tG'(t)/G(t) = \infty$ and $tG'(t)/t$ is increasing for large t. We also remark that the key argument in its proof is still a "localization principle", more general than (A), i.e.,

(B) for every $\xi \in \mathbb{M}$ and every $\{v_n\}_n \subset W^{1,p}(Y;\mathbb{R}^m)$ such that

$$\begin{cases} v_n \rightharpoonup l_\xi \text{ in } W^{1,p}(Y;\mathbb{R}^m) \\ \sup\limits_n \int_\Omega G(|\nabla v_n(x)|)dx < \infty, \end{cases}$$

there exist a subsequence $\{v_n\}_n$ (not relabeled) and $\{w_n\}_n \subset l_\xi + C_c^\infty(Y;\mathbb{R}^m)$ such that

$$\begin{cases} |\nabla v_n - \nabla w_n| \to 0 \text{ in measure} \\ \{G(|\nabla w_n|)\}_n \text{ is equi-integrable.} \end{cases}$$

It is easily seen that $(\mathrm{C}_{p,q})$ generalises (A) and (B) in a natural way, i.e.,

$$\begin{cases} \text{if } L \text{ has } p\text{-growth then (A) implies } (\mathrm{C}_{p,\infty}) \\ \text{if } L \text{ has fast growth then (B) implies } (\mathrm{C}_{p,\infty}), \end{cases}$$

which makes that Theorem 1.1 contains Acerbi-Fusco's theorem and Sychev's theorem in the homogeneous case.

The plan of the paper is as follows. Theorem 1.1 is proved in Section 3. Its proof uses some classical facts on Young measures that we recall in Section 2. (Note that it seems to be difficult to prove Theorem 1.1 without using Young measure theory.)

2. Some facts on Young measures

Young measures were introduced by Young in 1937 (see [You37]) with the purpose of finding an extension of the class of Sobolev functions for which one-dimensional nonconvex variational problems become solvable. In the context of the multidimensional calculus of variations, Kinderleherer and Pedregal (see [KP92, KP94]) and independently Kristensen (see [Kri94]) were the first to use Young measures for dealing with lower semicontinuity problems. Relaxation and convergence in energy problems were studied for the first time by Sychev via Young measures following a new approach to Young measures that he introduced in [Syc99]. In this section we only recall the ingredients that we need for proving Theorem 1.1. For more details on Young measure theory and its applications to the calculus of variations we refer to [Ped97, Ped00, Syc04].

Let $\mathcal{P}(\mathbb{M})$ be the set of all probability measures on $\mathbb{M}$, let $C(\mathbb{M})$ be the space of all continuous functions from $\mathbb{M}$ to $\mathbb{R}$ and let

$$C_0(\mathbb{M}) := \Big\{\Phi \in C(\mathbb{M}) : \lim_{|\xi|\to 0} \Phi(\xi) = 0\Big\}.$$

Here is the definition of a Young measure.

Definition 2.1. A family $(\mu_x)_{x\in\Omega}$ of probability measures on $\mathbb{M}$, i.e., $\mu_x \in \mathcal{P}(\mathbb{M})$ for all $x \in \Omega$, is said to be a Young measure if there exists a sequence $\{\xi_n\}_n$ of measurable functions from Ω to $\mathbb{M}$ such that

$$\Phi(\xi_n) \overset{*}{\rightharpoonup} \langle \Phi; \mu_{(\cdot)}\rangle \text{ in } L^\infty(\Omega) \text{ for all } \Phi \in C_0(\mathbb{M})$$

with $\langle \Phi; \mu_{(\cdot)}\rangle := \int_{\mathbb{M}} \Phi(\zeta) d\mu_{(\cdot)}(\zeta)$. In this case, we say that $\{\xi_n\}_n$ generates $(\mu_x)_{x\in\Omega}$ as a Young measure.

The following lemma makes clear the link between convergence in measure and Young measures. (The proof follows from the definition.)

Lemma 2.2. *let $\{\xi_n\}_n$ and $\{\zeta_n\}_n$ be two sequences of measurable functions from Ω to $\mathbb{M}$. If $\{\xi_n\}_n$ generates a Young measure and if $|\xi_n - \zeta_n| \to 0$ in measure then $\{\zeta_n\}_n$ generates the same Young measure.*

The following theorem gives a sufficient condition for proving the existence of Young measures (for a proof see [Bal89, Syc04, FL07]).

Theorem 2.3. *Let $\theta : \mathbb{M} \to \mathbb{R}$ be a continuous function such that $\lim_{|\zeta|\to\infty} \theta(\zeta) = \infty$ and let $\{\xi_n\}_n$ be a sequence of measurable functions from Ω to $\mathbb{M}$ such that*

$$\sup_n \int_\Omega \theta(\xi_n(x))dx < \infty.$$

Then, $\{\xi_n\}_n$ contains a subsequence generating a Young measure.

The following two theorems are important in dealing with integral functionals (for proofs see [Bal84, Syc99]).

Theorem 2.4 (semicontinuity theorem)**.** *Let $L : \mathbb{M} \to [0, \infty]$ be a continuous function and let $\{\xi_n\}_n$ be a sequence of measurable functions from Ω to $\mathbb{M}$ such that $\{\xi_n\}_n$ generates $(\mu_x)_{x\in\Omega}$ as a Young measure. Then*

$$\varliminf_{n\to\infty} \int_\Omega L(\xi_n(x))dx \geq \int_\Omega \langle L; \mu_x\rangle dx.$$

Theorem 2.5 (continuity theorem)**.** *Let $L : \mathbb{M} \to [0, \infty]$ be a continuous function and let $\{\xi_n\}_n$ be a sequence of measurable functions from Ω to $\mathbb{M}$ such that $\{\xi_n\}_n$ generates $(\mu_x)_{x\in\Omega}$ as a Young measure. Then*

$$\lim_{n\to\infty} \int_\Omega L(\xi_n(x))dx = \int_\Omega \langle L; \mu_x\rangle dx < \infty$$

if and only if $\{L(\xi_n)\}_n$ is equi-integrable.

3. Proof of Theorem 1.1

Let $\{u_n\}_n \subset W^{1,p}(\Omega; \mathbb{R}^m)$ and let $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ be such that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^m)$. We have to prove that

$$\varliminf_{n\to\infty} I(u_n) \geq I(u). \tag{3.1}$$

Step 1: localization. Without loss of generality we can assume that:

$$\|u_n - u\|_{L^p(\Omega;\mathbb{R}^m)} \to 0; \tag{3.2}$$

$$\infty > \varliminf_{n\to\infty} I(u_n) = \lim_{n\to\infty} I(u_n) \text{ and so } \sup_n \int_\Omega L(\nabla u_n(x))dx < \infty. \tag{3.3}$$

As $u_n \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^m)$ we have

$$\sup_n \int_\Omega |\nabla u_n(x)|^p dx < \infty, \tag{3.4}$$

and so, by Theorem 2.3, there exists a family $(\mu_x)_{x\in\Omega}$ of probability measures on $\mathbb{M}$ such that (up to a subsequence)

$$\{\nabla u_n\}_n \text{ generates } (\mu_x)_{x\in\Omega} \text{ as a Young measure.} \tag{3.5}$$

From Theorem 2.4 it follows that

$$\varliminf_{n\to\infty} I(u_n) \geq \int_\Omega \langle L; \mu_x\rangle dx$$

with (because (3.3) holds) for a.e. $x_0 \in \Omega$,

$$\langle L; \mu_{x_0}\rangle < \infty. \tag{3.6}$$

Thus, to prove (3.1) it is sufficient to show that for a.e. $x_0 \in \Omega$,

$$\langle L; \mu_{x_0}\rangle \geq L(\nabla u(x_0)). \tag{3.7}$$

Step 2: blow up. From (3.3) we deduce that there exist $f \in L^1(\Omega; [0, \infty[)$ and a finite positive Radon measure λ on Ω with $|\text{supp}(\lambda)| = 0$ such that (up to a subsequence) $L(\nabla u_n)dx \overset{*}{\rightharpoonup} f dx + \lambda$ in the sense of measures and for a.e. $x_0 \in \Omega$,

$$\lim_{r\to 0} \lim_{n\to\infty} \fint_{x_0+rY} L(\nabla u_n(x))dx = f(x_0) < \infty \tag{3.8}$$

with $Y :=]-\frac{1}{2}, \frac{1}{2}[^N$. By the same argument, from (3.4) we see that

$$\lim_{r\to 0} \lim_{n\to\infty} \fint_{x_0+rY} |\nabla u_n(x)|^p dx < \infty. \tag{3.9}$$

As $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ it follows that u is a.e. L^p-differentiable (see [Zie89, Theorem 3.4.2 p.129]), i.e., for a.e. $x_0 \in \Omega$,

$$\lim_{r\to 0} \frac{1}{r^{N+p}} \big\|u(x_0 + \cdot) - u(x_0) - \nabla u(x_0)y\big\|^p_{L^p(rY;\mathbb{R}^m)} = 0. \tag{3.10}$$

From (3.2) we see that (up to a subsequence) for a.e. $x_0 \in \Omega$,

$$|u_n(x_0) - u(x_0)|^p \to 0. \tag{3.11}$$

As $C_0(\mathbb{M})$ is separable we can assert that for a.e. $x_0 \in \Omega$, x_0 is a Lebesgue point of $\langle \Phi; \mu_{(\cdot)} \rangle$ for all $\Phi \in C_0(\mathbb{M})$, i.e.,

$$\lim_{r \to 0} \fint_{x_0 + rY} \langle \Phi, \mu_x \rangle dx = \langle \Phi, \mu_{x_0} \rangle \text{ for all } \Phi \in C_0(\mathbb{M}). \tag{3.12}$$

Fix any $x_0 \in \Omega$ such that (3.6), (3.8), (3.10), (3.11) and (3.12) hold and fix $r_0 > 0$ such that $x_0 + rY \subset \Omega$ for all $r \in]0, r_0]$. For each $n \geq 1$ and each $r \in]0, r_0]$, let $u_n^r \in W^{1,p}(Y; \mathbb{R}^m)$ and a family $(\mu_y^r)_{y \in Y}$ of probability measures on $\mathbb{M}$ be given by

$$\begin{cases} u_n^r(y) := \frac{1}{r}\left(u_n(x_0 + ry) - u_n(x_0)\right) \\ \mu_y^r := \mu_{x_0 + ry}. \end{cases}$$

Then (3.8) (resp. (3.9)) can be rewritten as

$$\lim_{r \to 0} \lim_{n \to \infty} \int_Y L(\nabla u_n^r(x)) dx < \infty \text{ (resp. } \lim_{r \to 0} \lim_{n \to \infty} \int_Y |\nabla u_n^r(x)|^p dx < \infty). \tag{3.13}$$

Taking (3.5) into account it is easy to see that for every $r \in]0, r_0]$, $\{\nabla u_n^r\}_n$ generates $(\mu_y^r)_{y \in Y}$ as a Young measure, i.e.,

$$\Phi(\nabla u_n^r) \overset{*}{\rightharpoonup} \langle \Phi, \mu_{(\cdot)}^r \rangle \text{ in } L^\infty(Y) \text{ as } n \to \infty \text{ for all } \Phi \in C_0(\mathbb{M}), \tag{3.14}$$

and using (3.12) it is clear that

$$\langle \Phi; \mu_{(\cdot)}^r \rangle \overset{*}{\rightharpoonup} \langle \Phi; \mu_{x_0} \rangle \text{ in } L^\infty(Y) \text{ as } r \to 0 \text{ for all } \Phi \in C_0(\mathbb{M}). \tag{3.15}$$

On the other hand, we have

$$\begin{aligned} \|u_{n,r} - l_{\nabla u(x_0)}\|_{L^p(Y;\mathbb{R}^m)}^p &= \int_Y |u_{n,r}(y) - l_{\nabla u(x_0)}(y)|^p dy \\ &= \frac{1}{r^{N+p}} \|u_n(x_0 + \cdot) - u_n(x_0) - l_{\nabla u(x_0)}\|_{L^p(rY;\mathbb{R}^m)}^p, \end{aligned}$$

and consequently

$$\begin{aligned} \|u_n^r - l_{\nabla u(x_0)}\|_{L^p(Y;\mathbb{R}^m)}^p &\leq \frac{c}{r^{N+p}} \|u_n - u\|_{L^p(\Omega;\mathbb{R}^m)}^p + \frac{c}{r^{N+p}} |u_n(x_0) - u(x_0)|^p \\ &\quad + \frac{c}{r^{N+p}} \|u(x_0 + \cdot) - u(x_0) - l_{\nabla u(x_0)}\|_{L^p(rY;\mathbb{R}^m)}^p \end{aligned}$$

with $c > 0$ which only depends on p. Using (3.2), (3.11) and (3.10) we deduce that

$$\lim_{r \to 0} \lim_{n \to \infty} \|u_n^r - l_{\nabla u(x_0)}\|_{L^p(Y;\mathbb{R}^m)} = 0. \tag{3.16}$$

According to (3.16), (3.13) and (3.14) together with (3.15), by diagonalization there exists a mapping $n \to r_n$ decreasing to 0 such that

$$\begin{cases} \|u_n^{r_n} - l_{\nabla u(x_0)}\|_{L^p(Y;\mathbb{R}^m)} \to 0 \\ \lim\limits_{n \to \infty} \int_Y |\nabla u_n^{r_n}(y)|^p dy < \infty, \text{ and so } \sup\limits_n \int_Y |\nabla u_n^{r_n}(y)|^p dy < \infty \\ \lim\limits_{n \to \infty} \int_Y L(\nabla u_n^{r_n}(y)) dy < \infty, \text{ and so } \sup\limits_n \int_Y L(\nabla u_n^{r_n}(y)) dy < \infty \\ \{\nabla u_n^{r_n}\}_n \text{ generates } \mu_{x_0} \text{ as a Young measure,} \end{cases}$$

and consequently we have:

$$\begin{cases} v_n \rightharpoonup l_{\nabla u(x_0)} \text{ in } W^{1,p}(Y;\mathbb{R}^m) \\ \sup\limits_n \int_Y L(\nabla v_n(y))dy < \infty; \end{cases} \tag{3.17}$$

$$\{\nabla v_n\}_n \text{ generates } \mu_{x_0} \text{ as a Young measure.} \tag{3.18}$$

where $v_n := u_n^{r_n}$.

Step 3: using ($\mathbf{C}_{p,q}$) and $W^{1,q}$-quasiconvexity. According to (3.17), by ($\mathrm{C}_{p,q}$) there exists $\{w_n\}_n \subset l_{\nabla u(x_0)} + W_0^{1,q}(Y;\mathbb{R}^m)$ such that

$$\begin{cases} |\nabla v_n - \nabla w_n| \to 0 \text{ in measure} \\ L(\nabla w_n) \text{ is equi-integrable,} \end{cases}$$

hence, by (3.18) and Lemma 2.2, $\{\nabla w_n\}_n$ generates μ_{x_0} as a Young measure, and, taking (3.6) into account, from Theorem 2.5 we deduce that

$$\lim_{n\to\infty} \int_Y L(\nabla w_n(y))dy = \langle L; \mu_{x_0}\rangle. \tag{3.19}$$

As L is $W^{1,q}$-quasiconvex, we have

$$\int_Y L(\nabla w_n(y))dy \geq L(\nabla u(x_0)) \text{ for all } n \geq 1,$$

and (3.7) follows by letting $n \to \infty$ and using (3.19). ■

Remark 3.1. In case $q = \infty$ the condition of $W^{1,q}$-quasiconvexity is the classical condition of quasiconvexity by Morrey (see [Mor52]).

Remark 3.2. In fact, we have also proved that if $\{u_n\}_n \subset W^{1,p}(\Omega;\mathbb{R}^m)$ is such that $\sup_n \int_\Omega L(\nabla u_n(x))dx < \infty$ and if $\{\nabla u_n\}_n$ generates $(\mu_x)_{x\in\Omega}$ as a Young measure, then for a.e. $x \in \Omega$, μ_x is a homogeneous gradient L-Young measure centered at $\nabla u(x)$, with $u \in W^{1,p}(\Omega;\mathbb{R}^m)$, provided that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$ and ($\mathrm{C}_{p,q}$) holds with $q \in [1,\infty]$. Homogeneous gradient L-Young measures were introduced and completely characterized by Sychev in [Syc00] where we refer the reader for more details.

Remark 3.3. From the proof of Theorem 1.1 we can extract the following lower semicontinuity theorem with the biting weak convergence.

Theorem 3.4. *Given $p \in]1,\infty[$ and $q \in [1,\infty]$, assume that L is $W^{1,q}$-quasiconvex and satisfies* ($\mathrm{C}_{p,q}$). *Then, for each $\{u_n\}_n \subset W^{1,p}(\Omega;\mathbb{R}^m)$ and each $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ such that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$ and $\sup_n \int_\Omega L(\nabla u_n(x))dx < \infty$, there exists a subsequence $\{u_n\}_n$ (not relabeled) and a family $(\mu_x)_{x\in\Omega}$ of probability measures on $\mathbb{M}$ such that*:

(i) $\{\nabla u_n\}_n$ *generates $(\mu_x)_{x\in\Omega}$ as a Young measure*;
(ii) $L(\nabla u_n) \overset{b}{\rightharpoonup} \langle L; \mu_{(\cdot)}\rangle$, *where "$\overset{b}{\rightharpoonup}$" denotes the biting weak convergence*;
(iii) $\langle L; \mu_x\rangle \geq L(\nabla u(x))$ *for a.a. $x \in \Omega$.*

For a deeper discussion of weak lower semicontinuity in the sense of biting lemma, see Ball and Zhang [BZ90] (see also [Syc05, Lemma 3.2] for a simple proof of the biting lemma).

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