

# Université de Nîmes

Laboratoire MIPA  
Université de Nîmes, Site des Carmes  
Place Gabriel Péri, 30021 Nîmes, France  
<http://mipa.unimes.fr>

## Localization principle and relaxation

by

JEAN-PHILIPPE MANDALLENA

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# LOCALIZATION PRINCIPLE AND RELAXATION

JEAN-PHILIPPE MANDALLENNA

ABSTRACT. Relaxation theorems for multiple integrals on  $W^{1,p}(\Omega; \mathbb{R}^m)$ , where  $p \in ]1, \infty[$ , are proved under general conditions on the integrand  $L : \mathbb{M} \rightarrow [0, \infty]$  which is Borel measurable and not necessarily finite. We involve a localization principle that we previously used to prove a general lower semicontinuity result.

## 1. INTRODUCTION

This paper is concerned with the problem of finding an integral representation for the “relaxed functional”  $\bar{I} : W^{1,p}(\Omega; \mathbb{R}^m) \rightarrow [0, \infty]$ , with  $p \in ]1, \infty[$ , given by

$$\bar{I}(u) = \inf \left\{ \liminf_{n \rightarrow \infty} \int_{\Omega} L(\nabla u_n(x)) dx : u_n \rightarrow u \text{ in } L^p(\Omega; \mathbb{R}^m) \right\}$$

with  $\Omega \subset \mathbb{R}^N$  a bounded open set with Lipschitz boundary and  $L : \mathbb{M} \rightarrow [0, \infty]$  a Borel measurable and  $p$ -coercive function, i.e.,  $L(\cdot) \geq C|\cdot|^p$  for some  $C > 0$ , where  $\mathbb{M}$  denotes the space of all real  $m \times N$  matrices.

The concept of “relaxed functional” was introduced by Bogolubov in 1930 in the case where  $m = N = 1$  (see [Bog30]). Bogolubov developed a one-dimensional theory of relaxation and highlighted the key role of the convexification of  $L$  for representing  $\bar{I}$ . Later, in 1974, Ekeland and Temam extended Bogolubov’s relaxation theory to the case  $N \geq 1$  and  $m = 1$  (see [ET74]). Then, in the vector case, i.e.,  $\min\{m, N\} > 1$ , in 1982, Dacorogna in [Dac82] (see also [Dac08]) proved that if  $L$  has  $p$ -growth, i.e.,  $L(\cdot) \leq c(1 + |\cdot|^p)$  for some  $c > 0$ , then  $\bar{I}$  can be represented by an integral whose integrand is given by the  $W^{1,\infty}$ -quasiconvexification, but not the convexification, of  $L$ , i.e.,  $\mathcal{Z}_{\infty}L : \mathbb{M} \rightarrow [0, \infty]$  defined by

$$\mathcal{Z}_{\infty}L(\xi) := \inf \left\{ \int_Y L(\nabla \phi(y)) dy : \phi \in l_{\xi} + W_0^{1,\infty}(Y; \mathbb{R}^m) \right\}$$

with  $Y := ]-\frac{1}{2}, \frac{1}{2}[^N$  and  $l_{\xi}(y) := \xi y$ . The paper of Dacorogna, i.e., [Dac82], is the point of departure of many works on the problem of representing  $\bar{I}$  by a variational integral in the vector case: several authors have tried to extend Dacorogna’s theorem to more general classes of integrands (see for instance [AF84, FM97, BB00, Syc05, AHM07, AHM08, AH10, Syc10, Syc]). In the same spirit, the object of the present paper is to study the existence of an integral representation of  $\bar{I}$  with respect to a general condition on  $L$ , that we called “localization principle”, i.e.,

$(C_{p,q})$  for every  $\xi \in \mathbb{M}$  and every  $\{v_n\}_n \subset W^{1,p}(Y; \mathbb{R}^m)$  such that

$$\begin{cases} v_n \rightarrow l_{\xi} \text{ in } L^p(Y; \mathbb{R}^m) \\ \sup_n \int_Y L(\nabla v_n(y)) dy < \infty, \end{cases}$$

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there exist a subsequence  $\{v_n\}_n$  (not relabeled) and a sequence  $\{w_n\}_n \subset l_\xi + W_0^{1,q}(Y; \mathbb{R}^m)$  such that

$$\begin{cases} |\nabla v_n - \nabla w_n| \rightarrow 0 \text{ in measure} \\ \{L(\nabla w_n)\}_n \text{ is equi-integrable.} \end{cases}$$

that we used in [Man] to prove the following lower semicontinuity theorem.

**Theorem 1.1.** *Let  $p \in ]1, \infty[$ , let  $q \in [1, \infty]$  and let  $L : \mathbb{M} \rightarrow [0, \infty]$  be a continuous and  $p$ -coercive function. If  $(C_{p,q})$  holds and if  $L$  is  $W^{1,q}$ -quasiconvex, i.e.,  $L = \mathcal{Z}_q L$  with  $\mathcal{Z}_q L : \mathbb{M} \rightarrow [0, \infty]$  given by*

$$\mathcal{Z}_q L(\xi) := \inf \left\{ \int_Y L(\nabla \phi(y)) dy : \phi \in l_\xi + W_0^{1,q}(Y; \mathbb{R}^m) \right\},$$

then the functional  $I : W^{1,p}(\Omega; \mathbb{R}^m) \rightarrow [0, \infty]$  defined by

$$I(u) := \int_\Omega L(\nabla u(x)) dx$$

is lower semicontinuous with respect to the strong convergence of  $L^p(\Omega; \mathbb{R}^m)$ .

Note that the condition  $(C_{p,q})$  is a generalization of previous ‘‘localization principles’’ (see [AF84, KP92, Syc99, Syc05]). Our goal here is to study how  $(C_{p,q})$  behave with respect to the relaxation.

In this paper we establish two extensions of Theorem 1.1 to the case of the relaxation (see Theorems 2.1 and 2.2). Roughly, the first theorem, i.e., Theorem 2.1, asserts that, when  $L$  is continuous on its effective domain and  $q \geq p$ , if  $(C_{p,q})$  holds and if  $W^{1,p}$ -sobolev functions can be approximated by piecewise affine functions both in  $L^p$ -norm and in  $\mathcal{Z}_q L$ -energy, see  $(H_{p,q})$  in the next section, then  $\bar{I}$  has an integral representation whose integrand is given the  $W^{1,q}$ -quasiconvexification of  $L$ , i.e., by  $\mathcal{Z}_q L$ . The second theorem, i.e., Theorem 2.2, which is a variant of the first one, says that if  $L$  is continuous on the interior of its effective domain and if  $q \geq p$ , then under additional assumptions related to the behavior of  $L$  on the boundary of its effective domain, see  $(R_1)$ ,  $(R_2)$  and  $(R_3)$  in the next section, and under slight modifications of  $(C_{p,q})$  and  $(H_{p,q})$ , see  $(\hat{C}_{p,q})$  and  $(\hat{H}_{p,q})$  in the next section, the functional  $\bar{I}$  can be represented by an integral whose integrand is given by the lower semicontinuous envelope of  $\mathcal{Z}_q L$  (see Remark 2.3). Note that, in our context, to obtain the full integral representations of  $\bar{I}$  in Theorems 2.1 and 2.2, it seems difficult to avoid assumption  $(H_{p,q})$  or its variant  $(\hat{H}_{p,q})$  (such a fact was also pointed out in [Syc]). However, partial integral representations of  $\bar{I}$  on piecewise affine functions can be established under only  $(C_{p,q})$  or its variant  $(\hat{C}_{p,q})$  together with  $(R_1)$ ,  $(R_2)$  and  $(R_3)$  (see Theorems 2.1(a) and 2.2(a)).

The plan of the paper is as follows. In §2 we state the main results, i.e., Theorems 2.1 and 2.2, that we prove respectively in §4 and §5. The proofs of Theorems 2.1(a) and 2.2(a) use in a fundamental way Young measure theory whose brief summary are given in §3.1. The proof of Theorem 2.2 also need the concept of radially uniformly upper semicontinuous integrand, recently introduced in [AH10], whose some facts are recalled in §3.2.

## 2. MAIN RESULTS

Let  $m, N \geq 1$  be two integers, let  $\Omega \subset \mathbb{R}^N$  be a bounded open set with Lipschitz boundary and let  $\mathbb{M}$  denote the space of all real  $m \times N$  matrices. Let  $p \in ]1, \infty[$ , let

$L : \mathbb{M} \rightarrow [0, \infty]$  be a Borel measurable function which is  $p$ -coercive, i.e.,  $L(\cdot) \geq C|\cdot|^p$  for some  $C > 0$ , and let  $I : W^{1,p}(\Omega; \mathbb{R}^m) \rightarrow [0, \infty]$  be defined by

$$I(u) := \int_{\Omega} L(\nabla u(x)) dx.$$

Denote the lower semicontinuous envelope (or the “relaxed functional”) of  $I$  with respect to the strong convergence in  $L^p(\Omega; \mathbb{R}^m)$  by  $\bar{I} : W^{1,p}(\Omega; \mathbb{R}^m) \rightarrow [0, \infty]$ , i.e.,

$$\bar{I}(u) := \inf \left\{ \varliminf_{n \rightarrow \infty} I(u_n) : u_n \rightarrow u \text{ in } L^p(\Omega; \mathbb{R}^m) \right\},$$

and for each  $q \in ]1, \infty]$ , define  $\mathcal{Z}_q L : \mathbb{M} \rightarrow [0, \infty]$  by

$$\mathcal{Z}_q L(\xi) := \inf \left\{ \int_Y L(\nabla \phi(y)) dy : \phi \in l_{\xi} + W_0^{1,q}(Y; \mathbb{R}^m) \right\},$$

where  $Y := ]-\frac{1}{2}, \frac{1}{2}[^N$  and  $l_{\xi}(y) := \xi y$ . Usually,  $\mathcal{Z}_q L$  is called the  $W^{1,q}$ -quasiconvexification of  $L$ . Given  $p \in ]1, \infty[$  and  $q \in ]1, \infty]$ , we consider two general conditions on  $L$ , i.e.,  $(C_{p,q})$  stated in the introduction and  $(H_{p,q})$  below. (In what follows,  $\text{Aff}(\Omega; \mathbb{R}^m)$  denotes the space of continuous piecewise affine functions from  $\Omega$  to  $\mathbb{R}^m$ .)

$(H_{p,q})$  For every  $u \in W^{1,p}(\Omega; \mathbb{R}^m) \setminus \text{Aff}(\Omega; \mathbb{R}^m)$  such that

$$\int_{\Omega} \mathcal{Z}_q L(\nabla u(x)) dx < \infty,$$

there exists  $\{u_n\}_n \subset \text{Aff}(\Omega; \mathbb{R}^m)$  such that

$$\begin{cases} u_n \rightarrow u \text{ in } L^p(\Omega; \mathbb{R}^m) \\ \varliminf_{n \rightarrow \infty} \int_{\Omega} \mathcal{Z}_q L(\nabla u_n(x)) dx \leq \int_{\Omega} \mathcal{Z}_q L(\nabla u(x)) dx. \end{cases}$$

Denote the effective domain of  $L$  by  $\mathbb{L}$ . The first main result of the paper is the following.

**Theorem 2.1.** *Consider  $p \in ]1, \infty[$  and  $q \in [p, \infty]$  and assume that  $L$  is continuous on  $\mathbb{L}$ .*

(a) *If  $(C_{p,q})$  holds then for every  $u \in \text{Aff}(\Omega; \mathbb{R}^m)$*

$$(2.1) \quad \bar{I}(u) = \int_{\Omega} \mathcal{Z}_q L(\nabla u(x)) dx$$

(b) *If  $(C_{p,q})$  and  $(H_{p,q})$  are satisfied then (2.1) holds for all  $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ .*

In order to take more general situations into account like singular behavior of  $L$  on  $\partial \mathbb{L}$  (see [AH10, AHMa, AHMb] and also [CDA02] for the scalar case) we are led to replace the assumption “ $L$  is continuous on  $\mathbb{L}$ ” by the weaker one “ $L$  is continuous on the interior of  $\mathbb{L}$ ”. In our context, this can be done by considering the three additional conditions  $(R_1)$ ,  $(R_2)$  and  $(R_3)$  below together with slight modifications of  $(C_{p,q})$  and  $(H_{p,q})$ , i.e.,  $(\hat{C}_{p,q})$  and  $(\hat{H}_{p,q})$  below.

$(R_1)$   $L$  is radially uniformly upper semicontinuous (ru-usc), i.e., there exists  $c > 0$  such that

$$\overline{\lim}_{t \rightarrow 1} \Delta_L^c(t) \leq 0$$

with  $\Delta_L^c : [0, 1] \rightarrow ]-\infty, \infty]$  given by

$$(2.2) \quad \Delta_L^c(t) := \sup_{\xi \in \mathbb{L}} \frac{L(t\xi) - L(\xi)}{c + L(\xi)}.$$

(The concept of ru-usc integrand was introduced by Anza Hafsa in [AH10], see also [AHMa, AHMb].) Note that if  $L$  is ru-usc, also is  $\mathcal{Z}_q L$  (see Proposition 3.8). In what follows, given any  $\mathbb{D} \subset \mathbb{M}$ ,  $\text{int}(\mathbb{D})$  (resp.  $\overline{\mathbb{D}}$ ) denotes the interior (resp. the closure) of  $\mathbb{D}$ .

(R<sub>2</sub>)  $t\overline{\mathbb{L}} \subset \text{int}(\mathbb{L})$  for all  $t \in ]0, 1[$ .

Denote the effective domain of  $\mathcal{Z}_q L$  by  $\mathcal{Z}_q \mathbb{L}$ .

(R<sub>3</sub>)  $t\overline{\mathcal{Z}_q \mathbb{L}} \subset \text{int}(\mathcal{Z}_q \mathbb{L})$  for all  $t \in ]0, 1[$ .

( $\widehat{\mathbb{C}}_{p,q}$ ) For every  $t \in ]0, 1[$ , every  $\xi \in \mathbb{M}$  and every  $\{v_n\}_n \subset W^{1,p}(Y; \mathbb{R}^m)$  such that

$$\begin{cases} v_n \rightarrow l_\xi \text{ in } L^p(Y; \mathbb{R}^m) \\ \sup_n \int_Y L(\nabla v_n(y)) dy < \infty \\ \nabla v_n(y) \in t\mathbb{L} \text{ for all } n \geq 1 \text{ and a.a. } y \in Y, \end{cases}$$

there exist a subsequence  $\{v_n\}_n$  (not relabeled) and a sequence  $\{w_n\}_n \subset l_\xi + W_0^{1,q}(Y; \mathbb{R}^m)$  such that

$$\begin{cases} |\nabla v_n - \nabla w_n| \rightarrow 0 \text{ in measure} \\ \{L(\nabla w_n)\}_n \text{ is equi-integrable} \\ \nabla w_n(y) \in \text{int}(\mathbb{L}) \text{ for all } n \geq 1 \text{ and a.a. } y \in Y. \end{cases}$$

( $\widehat{\mathbb{H}}_{p,q}$ ) For every  $t \in ]0, 1[$  and every  $u \in W^{1,p}(\Omega; \mathbb{R}^m) \setminus \text{Aff}(\Omega; \mathbb{R}^m)$  such that

$$\begin{cases} \int_\Omega \mathcal{Z}_q L(\nabla u(x)) dx < \infty \\ \nabla u(x) \in t\overline{\mathcal{Z}_q \mathbb{L}} \text{ for a.a. } x \in \Omega, \end{cases}$$

there exists  $\{u_n\}_n \subset \text{Aff}(\Omega; \mathbb{R}^m)$  such that

$$\begin{cases} u_n \rightarrow u \text{ in } L^p(\Omega; \mathbb{R}^m) \\ \overline{\lim}_{n \rightarrow \infty} \int_\Omega \mathcal{Z}_q L(\nabla u_n(x)) dx \leq \int_\Omega \mathcal{Z}_q L(\nabla u(x)) dx. \end{cases}$$

The second main result of the paper is the following.

**Theorem 2.2.** *Consider  $p \in ]1, \infty[$  and  $q \in [p, \infty]$  and assume that  $L$  is continuous on  $\text{int}(\mathbb{L})$  and (R<sub>1</sub>), (R<sub>2</sub>) and (R<sub>3</sub>) are verified.*

(a) *If ( $\widehat{\mathbb{C}}_{p,q}$ ) holds then for every  $u \in \text{Aff}(\Omega; \mathbb{R}^m)$*

$$(2.3) \quad \bar{I}(u) = \int_\Omega \widehat{\mathcal{Z}_q L}(\nabla u(x)) dx$$

*with  $\widehat{\mathcal{Z}_q L} : \mathbb{M} \rightarrow [0, \infty]$  given by*

$$(2.4) \quad \widehat{\mathcal{Z}_q L}(\xi) := \liminf_{t \rightarrow 1} \mathcal{Z}_q L(t\xi).$$

(b) *If ( $\widehat{\mathbb{C}}_{p,q}$ ) and ( $\widehat{\mathbb{H}}_{p,q}$ ) are satisfied then (2.3) holds for all  $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ .*

*Remark 2.3.* According to Lemma 3.9 and Theorem 3.7, we see that, in Theorem 2.2, the function  $\widehat{\mathcal{Z}_q L}$  is in fact the lower semicontinuous envelope of  $\mathcal{Z}_q L$  and has the representation

$$\widehat{\mathcal{Z}_q L}(\xi) = \begin{cases} \mathcal{Z}_q L(\xi) & \text{if } \xi \in \text{int}(\mathcal{Z}_q \mathbb{L}) \\ \lim_{t \rightarrow 1} \mathcal{Z}_q L(t\xi) & \text{if } \xi \in \partial(\mathcal{Z}_q \mathbb{L}) \\ \infty & \text{otherwise.} \end{cases}$$

### 3. AUXILIARY RESULTS

**3.1. Some facts on Young measures.** Young measures were introduced by Young in 1937 (see [You37]) with the purpose of finding an extension of the class of Sobolev functions for which one-dimensional nonconvex variational problems become solvable. In the context of the multidimensional calculus of variations, Kinderlehrer and Pedregal (see [KP92, KP94]) and independently Kristensen (see [Kri94]) were the first to use Young measures for dealing with lower semicontinuity problems. Relaxation and convergence in energy problems were studied for the first time by Sychev via Young measures following a new approach to Young measures that he introduced in [Syc99]. Here we only recall the ingredients that we need for proving Theorems 2.1 and 2.2. For more details on Young measure theory and its applications to the calculus of variations we refer to [Ped97, Ped00, Syc04].

Let  $\mathcal{P}(\mathbb{M})$  be the set of all probability measures on  $\mathbb{M}$ , let  $C(\mathbb{M})$  be the space of all continuous functions from  $\mathbb{M}$  to  $\mathbb{R}$  and let

$$C_0(\mathbb{M}) := \left\{ \Phi \in C(\mathbb{M}) : \lim_{|\xi| \rightarrow 0} \Phi(\xi) = 0 \right\}.$$

Here is the definition of a Young measure.

**Definition 3.1.** A family  $(\mu_x)_{x \in \Omega}$  of probability measures on  $\mathbb{M}$ , i.e.,  $\mu_x \in \mathcal{P}(\mathbb{M})$  for all  $x \in \Omega$ , is said to be a Young measure if there exists a sequence  $\{\xi_n\}_n$  of measurable functions from  $\Omega$  to  $\mathbb{M}$  such that

$$\Phi(\xi_n) \xrightarrow{*} \langle \Phi; \mu_{(\cdot)} \rangle \text{ in } L^\infty(\Omega) \text{ for all } \Phi \in C_0(\mathbb{M})$$

with  $\langle \Phi; \mu_{(\cdot)} \rangle := \int_{\mathbb{M}} \Phi(\zeta) d\mu_{(\cdot)}(\zeta)$ . In this case, we say that  $\{\xi_n\}_n$  generates  $(\mu_x)_{x \in \Omega}$  as a Young measure.

The following lemma makes clear the link between convergence in measure and Young measures. (The proof follows from the definition.)

**Lemma 3.2.** *let  $\{\xi_n\}_n$  and  $\{\zeta_n\}_n$  be two sequences of measurable functions from  $\Omega$  to  $\mathbb{M}$ . If  $\{\xi_n\}_n$  generates a Young measure and if  $|\xi_n - \zeta_n| \rightarrow 0$  in measure then  $\{\zeta_n\}_n$  generates the same Young measure.*

The following theorem gives a sufficient condition for proving the existence of Young measures (for a proof see [Bal89, Syc04, FL07]).

**Theorem 3.3.** *Let  $\theta : \mathbb{M} \rightarrow \mathbb{R}$  be a continuous function such that  $\lim_{|\zeta| \rightarrow \infty} \theta(\zeta) = \infty$  and let  $\{\xi_n\}_n$  be a sequence of measurable functions from  $\Omega$  to  $\mathbb{M}$  such that*

$$\sup_n \int_{\Omega} \theta(\xi_n(x)) dx < \infty.$$

*Then,  $\{\xi_n\}_n$  contains a subsequence generating a Young measure.*

The following two theorems are important in dealing with integral functionals (for proofs see [Bal84, Syc99]).

**Theorem 3.4** (semicontinuity theorem). *Let  $G : \mathbb{M} \rightarrow [0, \infty]$  be a Borel measurable function which is continuous on  $\mathbb{D} \subset \mathbb{M}$  and let  $\{\xi_n\}_n$  be a sequence of measurable functions from  $\Omega$  to  $\mathbb{M}$  such that*

$$\begin{cases} \xi_n(x) \in \mathbb{D} \text{ for all } n \geq 1 \text{ and a.a. } x \in \Omega; \\ \{\xi_n\}_n \text{ generates } (\mu_x)_{x \in \Omega} \text{ as a Young measure.} \end{cases}$$

Then

$$\liminf_{n \rightarrow \infty} \int_{\Omega} G(\xi_n(x)) dx \geq \int_{\Omega} \langle G; \mu_x \rangle dx.$$

**Theorem 3.5** (continuity theorem). *Let  $G : \mathbb{M} \rightarrow [0, \infty]$  be a Borel measurable function which is continuous on  $\mathbb{D} \subset \mathbb{M}$  and let  $\{\xi_n\}_n$  be a sequence of measurable functions from  $\Omega$  to  $\mathbb{M}$  such that*

$$\begin{cases} \xi_n(x) \in \mathbb{D} \text{ for all } n \geq 1 \text{ and a.a. } x \in \Omega; \\ \{\xi_n\}_n \text{ generates } (\mu_x)_{x \in \Omega} \text{ as a Young measure.} \end{cases}$$

Then

$$\lim_{n \rightarrow \infty} \int_{\Omega} G(\xi_n(x)) dx = \int_{\Omega} \langle G; \mu_x \rangle dx < \infty$$

if and only if  $\{G(\xi_n)\}_n$  is equi-integrable.

**3.2. Some facts on ru-usc integrands.** The concept of ru-usc integrand was introduced by Anza Hafsia in [AH10] to deal with relaxation of variational integrals in  $W^{1,\infty}$  with constraints on the gradient in the framework of the multidimensional calculus of variations (see also [Wag09]). In fact, in the scalar case, such constrained relaxation and homogenization problems was previously intensively studied by Carbone and De Arcangelis (see [CDA02]), but their techniques could not be generalized to the vector case. Recently, developing the concept of ru-usc integrand, we succeeded to deal with constraints on the gradient in the context of homogenization of multiple integrals in  $W^{1,p}$ , with  $p \in ]1, \infty]$ , in the vector case (see [AHMa, AHMb]). Here we only recall the ingredients that we need for proving Theorem 2.2. For more details on the concept of ru-usc integrand we refer to [AHMa, §3.1].

Let  $G : \mathbb{M} \rightarrow [0, \infty]$  be a Borel measurable function whose effective domain is denoted by  $\mathbb{G}$ . For each  $c > 0$  we define  $\Delta_G^c : [0, 1] \rightarrow ]-\infty, \infty]$  by

$$\Delta_G^c(t) := \sup_{\xi \in \mathbb{G}} \frac{G(t\xi) - G(\xi)}{c + G(\xi)}.$$

**Definition 3.6.** We say that  $G$  is radially uniformly upper semicontinuous (ru-usc) if there exists  $c > 0$  such that  $\lim_{t \rightarrow 1} \Delta_G^c(t) \leq 0$ .

Define  $\widehat{G} : \mathbb{M} \rightarrow [0, \infty]$  by

$$\widehat{G}(\xi) := \lim_{t \rightarrow 1} G(t\xi).$$

The interest of Definition 3.6 comes from the following theorem (for a proof see [AHMa]).

**Theorem 3.7.** *If  $G$  is ru-usc and*

$$(3.1) \quad t\overline{\mathbb{G}} \subset \text{int}(\mathbb{G}) \text{ for all } t \in ]0, 1[,$$



where  $\overline{\mathbb{G}}$  (resp.  $\text{int}(\mathbb{G})$ ) denotes the closure (resp. the interior) of  $\mathbb{G}$ , and  $G$  is lower semicontinuous (lsc) on  $\text{int}(\mathbb{G})$  then:

- (i)  $\widehat{G}(\xi) = \begin{cases} G(\xi) & \text{if } \xi \in \text{int}(\mathbb{G}) \\ \lim_{t \rightarrow 1} G(t\xi) & \text{if } \xi \in \partial\mathbb{G} \\ \infty & \text{otherwise;} \end{cases}$
- (ii)  $\widehat{G}$  is ru-usc;
- (iii)  $\widehat{G}$  is the lsc envelope of  $G$ .

Let  $q \in ]1, \infty]$  and let  $\mathcal{Z}_q G : \mathbb{M} \rightarrow [0, \infty]$  be given by

$$(3.2) \quad \mathcal{Z}_q G(\xi) := \inf \left\{ \int_Y G(\nabla \phi(y)) dy : \phi \in l_\xi + W_0^{1,q}(Y; \mathbb{R}^m) \right\}$$

with  $Y := ]-\frac{1}{2}, \frac{1}{2}[^N$  and  $l_\xi(y) := \xi y$ . The following proposition shows that ru-usc integrands have a nice behavior with respect to the  $W^{1,q}$ -quasiconvexification (for a proof see [AHMa]).

**Proposition 3.8.** *If  $G$  is ru-usc then  $\mathcal{Z}_q G$  is ru-usc.*

**3.3. Two properties of the  $W^{1,q}$ -quasiconvexification formula.** Here, we give two (classical) properties of the  $W^{1,q}$ -quasiconvexification formula, i.e.,  $\mathcal{Z}_q G : \mathbb{M} \rightarrow [0, \infty]$  defined by (3.2). Denote the effective domain of  $\mathcal{Z}_q G$  by  $\mathcal{Z}_q \mathbb{G}$ . Lemma 3.9 below is due to Fonseca (see [Fon88]).

**Lemma 3.9.**  *$\mathcal{Z}_q G$  is continuous on  $\text{int}(\mathcal{Z}_q \mathbb{G})$ .*

The proof of the following lemma can be found in [AHMa] (see also [AHM08, AHM07]).

**Lemma 3.10.** *Given  $\xi \in \mathbb{M}$  and a bounded open set  $U \subset \mathbb{R}^N$  there exists  $\{\phi_n\}_n \subset W_0^{1,q}(U; \mathbb{R}^m)$  such that*

$$\begin{cases} \lim_{n \rightarrow \infty} \|\phi_n\|_{L^q(U; \mathbb{R}^m)} = 0 \\ \lim_{n \rightarrow \infty} \int_U G(\xi + \nabla \phi_n(x)) dx = \mathcal{Z}_q G(\xi). \end{cases}$$

#### 4. PROOF OF THEOREM 2.1

In this section we prove Theorem 2.1

**4.1. Proof of Theorem 2.1(a).** We are going to prove the following two inequalities:

$$(4.1) \quad \bar{I}(u) \geq \int_\Omega \mathcal{Z}_q L(\nabla u(x)) dx \text{ for all } u \in W^{1,p}(\Omega; \mathbb{R}^m);$$

$$(4.2) \quad \bar{I}(u) \leq \int_\Omega \mathcal{Z}_q L(\nabla u(x)) dx \text{ for all } u \in \text{Aff}(\Omega; \mathbb{R}^m).$$

**Proof of (4.1).** Consider  $u \in W^{1,p}(\Omega; \mathbb{R}^m)$  and  $\{u_n\}_n \subset W^{1,p}(\Omega; \mathbb{R}^m)$  such that

$$(4.3) \quad \|u_n - u\|_{L^p(\Omega; \mathbb{R}^m)} \rightarrow 0,$$

and prove that

$$(4.4) \quad \varliminf_{n \rightarrow \infty} I(u_n) \geq \int_\Omega \mathcal{Z}_q L(\nabla u(x)) dx.$$

**Step 1: localization.** Without loss of generality we can assume that

$$(4.5) \quad \infty > \varliminf_{n \rightarrow \infty} I(u_n) = \lim_{n \rightarrow \infty} I(u_n) \text{ and so } \sup_n \int_{\Omega} L(\nabla u_n(x)) dx < \infty.$$

Hence  $\nabla u_n(x) \in \mathbb{L}$  for all  $n \geq 1$  and a.a.  $x \in \Omega$ , where  $\mathbb{L}$  denotes the effective domain of  $L$ . Since  $L$  is  $p$ -coercive, from (4.5) we see that  $\sup_n \int_{\Omega} |\nabla u_n(x)|^p dx < \infty$  and so, by Theorem 3.3, there exists a family  $(\mu_x)_{x \in \Omega}$  of probability measures on  $\mathbb{M}$  such that (up to a subsequence)

$$(4.6) \quad \{\nabla u_n\}_n \text{ generates } (\mu_x)_{x \in \Omega} \text{ as a Young measure.}$$

As  $L$  is continuous on  $\mathbb{L}$ , from Theorem 3.4 it follows that

$$\varliminf_{n \rightarrow \infty} I(u_n) \geq \int_{\Omega} \langle L; \mu_x \rangle dx$$

with (because (4.5) holds) for a.e.  $x_0 \in \Omega$ ,

$$(4.7) \quad \langle L; \mu_{x_0} \rangle < \infty.$$

Thus, to prove (4.4) it is sufficient to show that for a.e.  $x_0 \in \Omega$ ,

$$(4.8) \quad \langle L; \mu_{x_0} \rangle \geq \mathcal{Z}_q L(\nabla u(x_0)).$$

**Step 2: blow up.** From (4.5) we deduce that there exist  $f \in L^1(\Omega; [0, \infty[)$  and a finite positive Radon measure  $\lambda$  on  $\Omega$  with  $|\text{supp}(\lambda)| = 0$  such that (up to a subsequence)  $L(\nabla u_n) dx \xrightarrow{*} f dx + \lambda$  in the sense of measures and for a.e.  $x_0 \in \Omega$ ,

$$(4.9) \quad \lim_{r \rightarrow 0} \lim_{n \rightarrow \infty} \int_{x_0 + rY} L(\nabla u_n(x)) dx = f(x_0) < \infty$$

with  $Y := ]-\frac{1}{2}, \frac{1}{2}[^N$ . As  $u \in W^{1,p}(\Omega; \mathbb{R}^m)$  it follows that  $u$  is a.e.  $L^p$ -differentiable (see [Zie89, Theorem 3.4.2 p.129]), i.e., for a.e.  $x_0 \in \Omega$ ,

$$(4.10) \quad \lim_{r \rightarrow 0} \frac{1}{r^{N+p}} \|u(x_0 + \cdot) - u(x_0) - \nabla u(x_0)y\|_{L^p(rY; \mathbb{R}^m)}^p = 0.$$

From (4.3) we see that (up to a subsequence) for a.e.  $x_0 \in \Omega$ ,

$$(4.11) \quad |u_n(x_0) - u(x_0)|^p \rightarrow 0.$$

As  $C_0(\mathbb{M})$  is separable we can assert that for a.e.  $x_0 \in \Omega$ ,  $x_0$  is a Lebesgue point of  $\langle \Phi; \mu_{(\cdot)} \rangle$  for all  $\Phi \in C_0(\mathbb{M})$ , i.e.,

$$(4.12) \quad \lim_{r \rightarrow 0} \int_{x_0 + rY} \langle \Phi, \mu_x \rangle dx = \langle \Phi, \mu_{x_0} \rangle \text{ for all } \Phi \in C_0(\mathbb{M}).$$

Fix any  $x_0 \in \Omega$  such that (4.7), (4.9), (4.10), (4.11) and (4.12) hold and fix  $r_0 > 0$  such that  $x_0 + rY \subset \Omega$  for all  $r \in ]0, r_0]$ . For each  $n \geq 1$  and each  $r \in ]0, r_0]$ , let  $u_n^r \in W^{1,p}(Y; \mathbb{R}^m)$  and a family  $(\mu_y^r)_{y \in Y}$  of probability measures on  $\mathbb{M}$  be given by

$$\begin{cases} u_n^r(y) := \frac{1}{r} (u_n(x_0 + ry) - u_n(x_0)) \\ \mu_y^r := \mu_{x_0 + ry}. \end{cases}$$

Then (4.9) can be rewritten as

$$(4.13) \quad \lim_{r \rightarrow 0} \lim_{n \rightarrow \infty} \int_Y L(\nabla u_n^r(x)) dx < \infty.$$

Taking (4.6) into account it is easy to see that for every  $r \in ]0, r_0]$ ,  $\{\nabla u_n^r\}_n$  generates  $(\mu_y^r)_{y \in Y}$  as a Young measure, i.e.,

$$(4.14) \quad \Phi(\nabla u_n^r) \xrightarrow{*} \langle \Phi, \mu_{(\cdot)}^r \rangle \text{ in } L^\infty(Y) \text{ as } n \rightarrow \infty \text{ for all } \Phi \in C_0(\mathbb{M}),$$

and using (4.12) it is clear that

$$(4.15) \quad \langle \Phi, \mu_{(\cdot)}^r \rangle \xrightarrow{*} \langle \Phi, \mu_{x_0} \rangle \text{ in } L^\infty(Y) \text{ as } r \rightarrow 0 \text{ for all } \Phi \in C_0(\mathbb{M}).$$

On the other hand, we have

$$\begin{aligned} \|u_{n,r} - l_{\nabla u(x_0)}\|_{L^p(Y; \mathbb{R}^m)}^p &= \int_Y |u_{n,r}(y) - l_{\nabla u(x_0)}(y)|^p dy \\ &= \frac{1}{r^{N+p}} \|u_n(x_0 + \cdot) - u_n(x_0) - l_{\nabla u(x_0)}\|_{L^p(rY; \mathbb{R}^m)}^p, \end{aligned}$$

and consequently

$$\begin{aligned} \|u_n^r - l_{\nabla u(x_0)}\|_{L^p(Y; \mathbb{R}^m)}^p &\leq \frac{c}{r^{N+p}} \|u_n - u\|_{L^p(\Omega; \mathbb{R}^m)}^p + \frac{c}{r^{N+p}} |u_n(x_0) - u(x_0)|^p \\ &\quad + \frac{c}{r^{N+p}} \|u(x_0 + \cdot) - u(x_0) - l_{\nabla u(x_0)}\|_{L^p(rY; \mathbb{R}^m)}^p \end{aligned}$$

with  $c > 0$  which only depends on  $p$ . Using (4.3), (4.11) and (4.10) we deduce that

$$(4.16) \quad \lim_{r \rightarrow 0} \lim_{n \rightarrow \infty} \|u_n^r - l_{\nabla u(x_0)}\|_{L^p(Y; \mathbb{R}^m)} = 0.$$

According to (4.16), (4.13) and (4.14) together with (4.15), by diagonalization there exists a mapping  $n \rightarrow r_n$  decreasing to 0 such that:

$$(4.17) \quad \begin{cases} v_n \rightarrow l_{\nabla u(x_0)} \text{ in } L^p(Y; \mathbb{R}^m) \\ \lim_{n \rightarrow \infty} \int_Y L(\nabla v_n(y)) dy < \infty, \text{ and } \sup_n \int_Y L(\nabla v_n(y)) dy < \infty; \end{cases}$$

$$(4.18) \quad \{\nabla v_n\}_n \text{ generates } \mu_{x_0} \text{ as a Young measure.}$$

where  $v_n := u_n^{r_n}$ .

**Step 3: using  $(C_{p,q})$ .** According to (4.17), by  $(C_{p,q})$  there exists  $\{w_n\}_n \subset l_{\nabla u(x_0)} + W_0^{1,q}(Y; \mathbb{R}^m)$  such that

$$\begin{cases} |\nabla v_n - \nabla w_n| \rightarrow 0 \text{ in measure} \\ \mathbb{L}(\nabla w_n) \text{ is equi-integrable,} \end{cases}$$

hence, by (4.18) and Lemma 3.2,  $\{\nabla w_n\}_n$  generates  $\mu_{x_0}$  as a Young measure. In particular,  $\sup_n \int_Y L(\nabla w_n(y)) dy < \infty$ , and so  $\nabla w_n(y) \in \mathbb{L}$  for all  $n \geq 1$  and a.a.  $y \in Y$ . As  $L$  is continuous on  $\mathbb{L}$ , taking (4.7) into account, from Theorem 3.5 we deduce that

$$(4.19) \quad \lim_{n \rightarrow \infty} \int_Y L(\nabla w_n(y)) dy = \langle L; \mu_{x_0} \rangle.$$

On the other hand, by definition of  $\mathcal{Z}_q L$ , we see that

$$\int_Y L(\nabla w_n(y)) dy \geq \mathcal{Z}_q L(\nabla u(x_0)) \text{ for all } n \geq 1,$$

and (4.8) follows by letting  $n \rightarrow \infty$  and using (4.19). ■

*Remark 4.1.* Analyzing the step 2 of the proof of (4.1), it is easily seen that we have in fact proved the following lemma (that we will use in the proof of Theorem 2.2).

**Lemma 4.2.** *Let  $p \in ]1, \infty[$ , let  $G : \mathbb{M} \rightarrow [0, \infty]$  be a Borel measurable and  $p$ -coercive function, let  $\mathbb{D} \subset \mathbb{G}$ , where  $\mathbb{G}$  denotes the effective domain of  $G$ , let  $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ , let  $\{u_n\}_n \subset W^{1,p}(\Omega; \mathbb{R}^m)$  and let  $(\mu_x)_{x \in \Omega}$  be a family of probability measures on  $\mathbb{M}$ . Assume that*

$$\left\{ \begin{array}{l} u_n \rightarrow u \text{ in } L^p(\Omega; \mathbb{R}^m) \\ \sup_n \int_{\Omega} G(\nabla u_n(x)) dx < \infty \\ \nabla u_n(x) \in \mathbb{D} \text{ for all } n \geq 1 \text{ and a.a. } x \in \Omega \\ \{\nabla u_n\}_n \text{ generates } (\mu_x)_{x \in \Omega} \text{ as a Young measure.} \end{array} \right.$$

Then, for a.e.  $x_0 \in \Omega$ , there exists  $\{v_n\}_n \subset W^{1,p}(Y; \mathbb{R}^m)$  such that

$$\left\{ \begin{array}{l} v_n \rightarrow l_{\nabla u(x_0)} \text{ in } L^p(\Omega; \mathbb{R}^m) \\ \sup_n \int_Y G(\nabla v_n(x)) dx < \infty \\ \nabla v_n(y) \in \mathbb{D} \text{ for all } n \geq 1 \text{ and a.a. } y \in Y \\ \{\nabla v_n\}_n \text{ generates } \mu_{x_0} \text{ as a Young measure.} \end{array} \right.$$

**Proof of (4.2).** Given  $u \in \text{Aff}(\Omega; \mathbb{R}^m)$  there exists a finite family  $\{U_i\}_{i \in I}$  of open disjoint subsets of  $\Omega$  such that  $|\Omega \setminus \cup_{i \in I} U_i| = 0$  and, for each  $i \in I$ ,  $|\partial U_i| = 0$  and  $\nabla u(x) = \xi_i$  in  $U_i$  with  $\xi_i \in \mathbb{M}$ . Thus

$$(4.20) \quad \int_{\Omega} \mathcal{Z}_q L(\nabla u(x)) dx = \sum_{i \in I} |U_i| \mathcal{Z}_q L(\xi_i).$$

Recalling that  $q \geq p$  and using Lemma 3.10, for each  $i \in I$ , we can assert that there exists  $\{\phi_n^i\}_n \subset W_0^{1,p}(U_i; \mathbb{R}^m)$  such that:

$$(4.21) \quad \lim_{n \rightarrow \infty} \|\phi_n^i\|_{L^p(U_i; \mathbb{R}^m)} = 0;$$

$$(4.22) \quad \lim_{n \rightarrow \infty} \int_{U_i} L(\xi_i + \nabla \phi_n^i(x)) dx = \mathcal{Z}_q L(\xi_i).$$

Define  $\{u_n\}_n \subset W^{1,p}(\Omega; \mathbb{R}^m)$  by

$$u_n(x) := u(x) + \phi_n^i(x) \text{ if } x \in U_i.$$

Using (4.21) it is easy to see that  $\|u_n - u\|_{L^p(\Omega; \mathbb{R}^m)} \rightarrow 0$ , and combining (4.22) with (4.20) we deduce that

$$\lim_{n \rightarrow \infty} \int_{\Omega} L(\nabla u_n(x)) dx = \int_{\Omega} \mathcal{Z}_q L(\nabla u(x)) dx,$$

and the result follows.  $\blacksquare$

*Remark 4.3.* Analyzing the previous proof, it is easily seen that we have in fact proved the following lemma.

**Lemma 4.4.** *Let  $p \in ]1, \infty[$  and  $q \in [p, \infty]$  and let  $G : \mathbb{M} \rightarrow [0, \infty]$  be a Borel measurable function. For every  $u \in \text{Aff}(\Omega; \mathbb{R}^m)$  there exists  $\{u_n\}_n \subset W^{1,p}(\Omega; \mathbb{R}^m)$  such that*

$$\left\{ \begin{array}{l} \lim_{n \rightarrow \infty} \|u_n - u\|_{L^p(\Omega; \mathbb{R}^m)} = 0 \\ \lim_{n \rightarrow \infty} \int_{\Omega} G(\nabla u_n(x)) dx = \int_{\Omega} \mathcal{Z}_q G(\nabla u(x)) dx. \end{array} \right.$$

4.2. **Proof of Theorem 2.1(b).** It is sufficient to prove that

$$\bar{I}(u) \leq \int_{\Omega} \mathcal{Z}_q L(\nabla u(x)) dx \text{ for all } u \in W^{1,p}(\Omega; \mathbb{R}^m) \setminus \text{Aff}(\Omega; \mathbb{R}^m).$$

Let  $u \in W^{1,p}(\Omega; \mathbb{R}^m) \setminus \text{Aff}(\Omega; \mathbb{R}^m)$  be such that  $\int_{\Omega} \mathcal{Z}_q L(\nabla u(x)) dx < \infty$ . By  $(H_{p,q})$  there exists  $\{u_k\}_k \subset \text{Aff}(\Omega; \mathbb{R}^m)$  such that

$$(4.23) \quad \begin{cases} \lim_{k \rightarrow \infty} \|u_k - u\|_{L^p(\Omega; \mathbb{R}^m)} = 0 \\ \overline{\lim}_{k \rightarrow \infty} \int_{\Omega} \mathcal{Z}_q L(\nabla u_k(x)) dx \leq \int_{\Omega} \mathcal{Z}_q L(\nabla u(x)) dx. \end{cases}$$

From Lemma 4.4 we deduce that for every  $k \geq 1$ , there exists  $\{u_{n,k}\}_n \subset W^{1,p}(\Omega; \mathbb{R}^m)$  such that

$$(4.24) \quad \begin{cases} \lim_{n \rightarrow \infty} \|u_{n,k} - u_k\|_{L^p(\Omega; \mathbb{R}^m)} = 0 \\ \lim_{n \rightarrow \infty} \int_{\Omega} L(\nabla u_{n,k}(x)) dx = \int_{\Omega} \mathcal{Z}_q L(\nabla u_k(x)) dx. \end{cases}$$

Combining (4.24) with (4.23), we conclude that

$$\begin{cases} \overline{\lim}_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \|u_{n,k} - u\|_{L^p(\Omega; \mathbb{R}^m)} = 0 \\ \overline{\lim}_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\Omega} L(\nabla u_{n,k}(x)) dx \leq \int_{\Omega} \mathcal{Z}_q L(\nabla u(x)) dx. \end{cases}$$

and the result follows by diagonalization. ■

## 5. PROOF OF THEOREM 2.2

In this section we prove Theorem 2.2.

5.1. **Proof of Theorem 2.2(a).** We are going to prove the following two inequalities:

$$(5.1) \quad \bar{I}(u) \geq \int_{\Omega} \widehat{\mathcal{Z}_q L}(\nabla u(x)) dx \text{ for all } u \in W^{1,p}(\Omega; \mathbb{R}^m);$$

$$(5.2) \quad \bar{I}(u) \leq \int_{\Omega} \widehat{\mathcal{Z}_q L}(\nabla u(x)) dx \text{ for all } u \in \text{Aff}(\Omega; \mathbb{R}^m).$$

**Proof of (5.1).** Consider  $u \in W^{1,p}(\Omega; \mathbb{R}^m)$  and  $\{u_n\}_n \subset W^{1,p}(\Omega; \mathbb{R}^m)$  such that

$$(5.3) \quad \|u_n - u\|_{L^p(\Omega; \mathbb{R}^m)} \rightarrow 0,$$

and prove that

$$(5.4) \quad \underline{\lim}_{n \rightarrow \infty} I(u_n) \geq \int_{\Omega} \widehat{\mathcal{Z}_q L}(\nabla u(x)) dx.$$

Without loss of generality we can assume that

$$(5.5) \quad \infty > \underline{\lim}_{n \rightarrow \infty} I(u_n) = \lim_{n \rightarrow \infty} I(u_n) \text{ and so } \sup_n \int_{\Omega} L(\nabla u_n(x)) dx < \infty.$$

**Step 1: using ru-usc assumption.** According to the definition of  $\widehat{\mathcal{Z}}_q L$ , see (2.4), and Fatou's lemma, to establish (5.4) it is sufficient to show that

$$(5.6) \quad \varliminf_{n \rightarrow \infty} I(u_n) \geq \int_{\Omega} \mathcal{Z}_q L(t \nabla u(x)) dx.$$

for all  $t \in ]0, 1[$ . On the other hand, for any  $c > 0$ ,

$$(5.7) \quad \int_{\Omega} L(t \nabla u_n(x)) dx \leq (1 + \Delta_L^c(t)) \int_{\Omega} L(\nabla u_n(x)) dx + c |\Omega| \Delta_L^c(t)$$

for all  $n \geq 1$  and all  $t \in ]0, 1[$ , where  $\Delta_L^c(t)$  is given by (2.2), and consequently

$$\varliminf_{t \rightarrow 1} \varliminf_{n \rightarrow \infty} \int_{\Omega} L(t \nabla u_n(x)) dx \leq \varliminf_{n \rightarrow \infty} \int_{\Omega} L(\nabla u_n(x)) dx$$

because  $L$  is ru-usc, i.e.,  $\overline{\lim}_{t \rightarrow 1} \Delta_L^c(t) \leq 0$  for some  $c > 0$ . Hence, we are reduced to prove that for every  $t \in ]0, 1[$ ,

$$(5.8) \quad \varliminf_{n \rightarrow \infty} \int_{\Omega} L(t \nabla u_n(x)) dx \geq \int_{\Omega} \mathcal{Z}_q L(t \nabla u(x)) dx.$$

**Step 2: localization.** Fix  $t \in ]0, 1[$ . Using (5.5) and (5.7) we see that

$$(5.9) \quad \sup_n \int_{\Omega} L(t \nabla u_n(x)) dx < \infty.$$

Since  $L$  is  $p$ -coercive,  $\sup_n \int_{\Omega} |t \nabla u_n(x)|^p dx < \infty$  by (5.9). From Theorem 3.3 we deduce that there exists a family  $(\mu_x)_{x \in \Omega}$  of probability measures on  $\mathbb{M}$  such that (up to a subsequence)

$$(5.10) \quad \{t \nabla u_n\}_n \text{ generates } (\mu_x)_{x \in \Omega} \text{ as a Young measure.}$$

On the other hand, (5.5) implies that for every  $n \geq 1$  and a.e.  $x \in \Omega$ ,  $\nabla u_n(x) \in \mathbb{L}$ , hence

$$(5.11) \quad t \nabla u_n(x) \in t \mathbb{L} \text{ for all } n \geq 1 \text{ and a.a. } x \in \Omega.$$

Since  $t \overline{\mathbb{L}} \subset \text{int}(\mathbb{L})$ , where  $\text{int}(\mathbb{L})$  denotes the interior of  $\mathbb{L}$ , for every  $n \geq 1$  and a.e.  $x \in \Omega$ ,  $t \nabla u_n(x) \in \text{int}(\mathbb{L})$ . As  $L$  is continuous on  $\text{int}(\mathbb{L})$ , from Theorem 3.4 it follows that

$$\varliminf_{n \rightarrow \infty} \int_{\Omega} L(t \nabla u_n(x)) dx \geq \int_{\Omega} \langle L; \mu_x \rangle dx$$

with (because (5.9) holds) for a.e.  $x_0 \in \Omega$ ,

$$(5.12) \quad \langle L; \mu_{x_0} \rangle < \infty.$$

Thus, to prove (5.8) it is sufficient to show that for a.e.  $x_0 \in \Omega$ ,

$$(5.13) \quad \langle L; \mu_{x_0} \rangle \geq \mathcal{Z}_q L(t \nabla u(x_0)).$$

**Step 3: using Lemma 4.2 and  $(\widehat{\mathbf{C}}_{p,q})$ .** By (5.3) it is clear that  $tu_n \rightarrow tu$  in  $L^p(\Omega; \mathbb{R}^m)$ . Taking (5.9), (5.10) and (5.11) into account, from Lemma 4.2 we deduce that for a.e.  $x_0 \in \Omega$ , there exists  $\{v_n\}_n \subset W^{1,p}(Y; \mathbb{R}^m)$  such that:

$$(5.14) \quad \begin{cases} v_n \rightarrow l_{t\nabla u(x_0)} \text{ in } L^p(\Omega; \mathbb{R}^m) \\ \sup_n \int_Y L(\nabla v_n(x)) dx < \infty \\ \nabla v_n(y) \in t\mathbb{L} \text{ for all } n \geq 1 \text{ and a.a. } y \in Y; \end{cases}$$

$$(5.15) \quad \{\nabla v_n\}_n \text{ generates } \mu_{x_0} \text{ as a Young measure.}$$

Fix any  $x_0 \in \Omega$  such that (5.14) and (5.15) hold. By  $(\widehat{\mathbf{C}}_{p,q})$  there exists  $\{w_n\}_n \subset l_{t\nabla u(x_0)} + W_0^{1,q}(Y; \mathbb{R}^m)$  such that

$$\begin{cases} |\nabla v_n - \nabla w_n| \rightarrow 0 \text{ in measure} \\ L(\nabla w_n) \text{ is equi-integrable} \\ \nabla w_n(y) \in \text{int}(\mathbb{L}) \text{ for all } n \geq 1 \text{ and a.a. } y \in Y, \end{cases}$$

hence, by (5.15) and Lemma 3.2,  $\{\nabla w_n\}_n$  generates  $\mu_{x_0}$  as a Young measure. As  $L$  is continuous on  $\text{int}(\mathbb{L})$ , taking (5.12) into account, from Theorem 3.5 we deduce that

$$(5.16) \quad \lim_{n \rightarrow \infty} \int_Y L(\nabla w_n(y)) dy = \langle L; \mu_{x_0} \rangle.$$

On the other hand, by definition of  $\mathcal{Z}_q L$ , we see that

$$\int_Y L(\nabla w_n(y)) dy \geq \mathcal{Z}_q L(t\nabla u(x_0)) \text{ for all } n \geq 1,$$

and (5.13) follows by letting  $n \rightarrow \infty$  and using (5.16).  $\blacksquare$

**Proof of (5.2).** Let  $u \in \text{Aff}(\Omega; \mathbb{R}^m)$  be such that  $\int_\Omega \widehat{\mathcal{Z}}_q L(\nabla u(x)) dx < \infty$ . Then

$$(5.17) \quad \nabla u(x) \in \widehat{\mathcal{Z}}_q \mathbb{L} \text{ for all } n \geq 1 \text{ and a.a. } x \in \Omega,$$

where  $\widehat{\mathcal{Z}}_q \mathbb{L}$  denotes the effective domain of  $\widehat{\mathcal{Z}}_q L$ . Since  $L$  is ru-usc, also is  $\mathcal{Z}_q L$ . Moreover, by assumption,

$$(5.18) \quad t\overline{\mathcal{Z}}_q \mathbb{L} \subset \text{int}(\mathcal{Z}_q \mathbb{L}) \text{ for all } t \in ]0, 1[,$$

where  $\overline{\mathcal{Z}}_q \mathbb{L}$  (resp.  $\text{int}(\mathcal{Z}_q \mathbb{L})$ ) denotes the closure (resp. the interior) of  $\mathcal{Z}_q \mathbb{L}$ , and  $\mathcal{Z}_q L$  is continuous on  $\text{int}(\mathcal{Z}_q \mathbb{L})$  by Lemma 3.9. From Theorem 3.7 it follows that:

$$(5.19) \quad \widehat{\mathcal{Z}}_q L(\xi) = \begin{cases} \mathcal{Z}_q L(\xi) & \text{if } \xi \in \text{int}(\mathcal{Z}_q \mathbb{L}) \\ \lim_{t \rightarrow 1} \mathcal{Z}_q L(t\xi) & \text{if } \xi \in \partial(\mathcal{Z}_q \mathbb{L}) \\ \infty & \text{otherwise;} \end{cases}$$

$$(5.20) \quad \widehat{\mathcal{Z}}_q L \text{ is ru-usc, i.e., } \overline{\lim}_{t \rightarrow 1} \Delta_{\widehat{\mathcal{Z}}_q L}^c(t) \leq 0 \text{ for some } c > 0.$$

By (5.19) we have  $\widehat{\mathcal{Z}}_q \mathbb{L} \subset \overline{\mathcal{Z}}_q \mathbb{L}$ , and so  $t\nabla u(x) \in \text{int}(\mathcal{Z}_q \mathbb{L})$  for all  $t \in ]0, 1[$  because of (5.17) and (5.18). Thus

$$\int_\Omega \mathcal{Z}_q L(t\nabla u(x)) dx \leq (1 + \Delta_{\widehat{\mathcal{Z}}_q L}^c(t)) \int_\Omega \widehat{\mathcal{Z}}_q L(\nabla u(x)) dx + c|\Omega| \Delta_{\widehat{\mathcal{Z}}_q L}^c(t)$$

for all  $t \in ]0, 1[$ , and consequently

$$(5.21) \quad \overline{\lim}_{t \rightarrow 1} \int_\Omega \mathcal{Z}_q L(t\nabla u(x)) dx \leq \int_\Omega \widehat{\mathcal{Z}}_q L(\nabla u(x)) dx$$

because (5.20) holds. On the other hand, it is clear that

$$(5.22) \quad \lim_{t \rightarrow 1} \|tu - u\|_{L^p(\Omega; \mathbb{R}^m)} = 0.$$

As  $u \in \text{Aff}(\Omega; \mathbb{R}^m)$  we have  $tu \in \text{Aff}(\Omega; \mathbb{R}^m)$  for all  $t \in ]0, 1[$ . From Lemma 4.4 we deduce that for each  $t \in ]0, 1[$ , there exists  $\{u_{n,t}\}_n \subset W^{1,p}(\Omega; \mathbb{R}^m)$  such that

$$\begin{cases} \lim_{n \rightarrow \infty} \|u_{n,t} - tu\|_{L^p(\Omega; \mathbb{R}^m)} = 0 \\ \lim_{n \rightarrow \infty} \int_{\Omega} L(\nabla u_{n,t}(x)) dx = \int_{\Omega} \mathcal{Z}_q L(t\nabla u(x)) dx, \end{cases}$$

and so

$$\begin{cases} \lim_{t \rightarrow 1} \lim_{n \rightarrow \infty} \|u_{n,t} - u\|_{L^p(\Omega; \mathbb{R}^m)} = 0 \\ \overline{\lim}_{t \rightarrow 1} \lim_{n \rightarrow \infty} \int_{\Omega} L(\nabla u_{n,t}(x)) dx \leq \int_{\Omega} \widehat{\mathcal{Z}}_q L(\nabla u(x)) dx \end{cases}$$

by using (5.22) and (5.21), and the result follows by diagonalization. ■

**5.2. Proof of Theorem 2.2(b).** It is sufficient to prove that

$$\bar{I}(u) \leq \int_{\Omega} \widehat{\mathcal{Z}}_q L(\nabla u(x)) dx \text{ for all } u \in W^{1,p}(\Omega; \mathbb{R}^m) \setminus \text{Aff}(\Omega; \mathbb{R}^m).$$

Let  $u \in W^{1,p}(\Omega; \mathbb{R}^m) \setminus \text{Aff}(\Omega; \mathbb{R}^m)$  be such that  $\int_{\Omega} \widehat{\mathcal{Z}}_q L(\nabla u(x)) dx < \infty$ . Arguing as in the proof of the inequality (5.2), we have (5.22) and (5.21) and for every  $t \in ]0, 1[$ ,

$$\begin{cases} \int_{\Omega} \mathcal{Z}_q L(t\nabla u(x)) dx < \infty \\ t\nabla u(x) \in t\overline{\mathcal{Z}}_q \mathbb{L} \text{ for a.a. } x \in \Omega. \end{cases}$$

Fix any  $t \in ]0, 1[$ . By  $(\widehat{H}_{p,q})$  there exists  $\{u_{k,t}\}_k \subset \text{Aff}(\Omega; \mathbb{R}^m)$  such that

$$(5.23) \quad \begin{cases} \lim_{k \rightarrow \infty} \|u_{k,t} - tu\|_{L^p(\Omega; \mathbb{R}^m)} = 0 \\ \overline{\lim}_{k \rightarrow \infty} \int_{\Omega} \mathcal{Z}_q L(\nabla u_{k,t}(x)) dx \leq \int_{\Omega} \mathcal{Z}_q L(t\nabla u(x)) dx. \end{cases}$$

Fix any  $k \geq 1$ . By Lemma 4.4 there exists  $\{u_{n,k,t}\}_n \subset W^{1,p}(\Omega; \mathbb{R}^m)$  such that

$$(5.24) \quad \begin{cases} \lim_{n \rightarrow \infty} \|u_{n,k,t} - u_{k,t}\|_{L^p(\Omega; \mathbb{R}^m)} = 0 \\ \lim_{n \rightarrow \infty} \int_{\Omega} L(\nabla u_{n,k,t}(x)) dx = \int_{\Omega} \mathcal{Z}_q L(\nabla u_{k,t}(x)) dx. \end{cases}$$

Combining (5.24), (5.23) with (5.22) together with (5.21), we conclude that

$$\begin{cases} \overline{\lim}_{t \rightarrow 1} \overline{\lim}_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \|u_{n,k,t} - u\|_{L^p(\Omega; \mathbb{R}^m)} = 0 \\ \overline{\lim}_{t \rightarrow 1} \overline{\lim}_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\Omega} L(\nabla u_{n,k,t}(x)) dx \leq \int_{\Omega} \widehat{\mathcal{Z}}_q L(\nabla u(x)) dx. \end{cases}$$

and the result follows by diagonalization. ■

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LABORATOIRE MIPA (MATHÉMATIQUES, INFORMATIQUE, PHYSIQUE ET APPLICATIONS)  
UNIVERSITE DE NIMES, SITE DES CARMES, PLACE GABRIEL PÉRI, 30021 NÎMES, FRANCE.  
*E-mail address:* `jean-philippe.mandalenna@unimes.fr`