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## Nonlinear capacitary problems for a non-periodic distribution of fibers

by

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October 2011

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October 14, 2011

**Abstract**

We determine the effective electric properties of a composite with high contrast. The energy density is given locally in terms of a convex function of the gradient of the potential. The permittivity may take very large values in a fairly general distribution of parallel fibers of tiny cross sections. For a critical size of the cross sections, we show that a concentration of electric energy may arise in a small region of space surrounding the fibers. This extra contribution is caused by the discrepancy between the behaviors of the potential in the matrix and in the fibers and is characterized by the density of the cross sections of the fibers with respect to the cross section of the body in terms of some suitable notion of capacity. Our results extend those established in [5] in the periodic case for the $p$-Laplacian to a general nonlinear framework and a non-periodic distribution of fibers.

AMS *subject classifications*: 32U20, 49J45, 74Q05.

*Keywords:* asymptotic analysis, Γ-convergence, capacity theory, homogenization, fibered media.

# Contents



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# 1 Introduction and setting out of the problem

Composites comprising traces of materials with extreme physical properties have been investigated by several authors over the past decades in various contexts, such as diffusion equations [5], [9], [11], [15], [17], fluid mechanics [10], electromagnetic theory [7], linearized elasticity [4], [6]. The common feature of this body of work is the emergence of a concentration of energy in a small region of space surrounding the strong components. This extra contribution is characterized by a local density of the geometric perturbations in terms of an appropriate capacity depending on the type of equations.

In this paper, we determine the effective electric properties of an electrified composite whereby a set of extremely thin fibers with very large permittivities is embedded in a matrix with permittivity of order 1. This study may as well concern various steady-state situations in Physics like heat diffusion for instance. It is interesting to refer to Electricity where capacity has a specific meaning. A similar problem has been studied by one of the authors with G. Bouchitté [5] in the periodic quasilinear case, for fibers of circular cross section. In what follows, we investigate the non periodic case and consider a more general non linear framework and also fibers with arbitrarily shaped cross sections. This is worthwhile, because fibers stem from draw plates, and therefore are likely to display anisotropic behaviors governed by general convex functions. Dropping the assumption of periodicity is a challenging task which may lead to quite different effective problems when composites with high contrast are considered. In our specific study, the effective problem turns out to show the same general features as in the periodic case, provided the fibers are not too closely spaced (see (1.6)).

We turn now to a more detailed introduction of the paper. Let $\mathcal{O} = \widehat{\mathcal{O}} \times (0, L)$ be a bounded smooth cylindrical open subset of $\mathbb{R}^3$. We consider the boundary value problem in Electrostatics

$$(\mathcal{P}_\varepsilon) \quad \begin{cases} \displaystyle\min_{u \in u_0 + W^{1,p}_{\Gamma_0}(\mathcal{O})} F_\varepsilon(u) - \int_{\mathcal{O}} q_b u \, dx - \int_{\Gamma_1} q_s u \, d\mathcal{H}^2, \\ W^{1,p}_{\Gamma_0}(\mathcal{O}) = \left\{\varphi \in W^{1,p}(\mathcal{O}) : \varphi = 0 \text{ on } \Gamma_0\right\}, \quad \Gamma_0 \subset \partial\mathcal{O}, \quad \mathcal{H}^2(\Gamma_0) > 0, \quad \Gamma_1 = \partial\mathcal{O} \setminus \Gamma_0, \\ (q_b, q_s) \in L^{p'}(\mathcal{O}) \times L^{p'}(\Gamma_1), \quad u_0 \in C^1(\overline{\mathcal{O}}), \quad \left(\dfrac{1}{p} + \dfrac{1}{p'} = 1\right), \\ F_\varepsilon(u) = \displaystyle\int_{\mathcal{O} \setminus T_{r_\varepsilon}} f(\nabla u) \, dx + \lambda_\varepsilon \int_{T_{r_\varepsilon}} g(\nabla u) \, dx. \end{cases} \tag{1.1}$$

The solution $u_\varepsilon$ of $(\mathcal{P}_\varepsilon)$ describes the electric potential of an electrified fibered composite insulator, where the distributions of body and surface charges are denoted by $q_b$ and $q_s$. The non periodic set $T_{r_\varepsilon}$ occupied by the fibers is defined in terms of a bounded domain $S \subset \mathbb{R}^2$ with a Lipschitz boundary, of two small positive parameter $\varepsilon, r_\varepsilon$ such that $0 < r_\varepsilon \ll \varepsilon \ll 1$, and of a finite set

$$\Omega_\varepsilon = \left\{\omega^j_\varepsilon, \ j \in J_\varepsilon\right\} \subset \widehat{\mathcal{O}}, \quad J_\varepsilon \subset \mathbb{N}, \tag{1.2}$$

by setting

$$T_{r_\varepsilon} := \bigcup_{j \in J_\varepsilon} T^j_{r_\varepsilon}, \quad T^j_{r_\varepsilon} := (\omega^j_\varepsilon + r_\varepsilon S) \times (0, L). \tag{1.3}$$

The parameter $r_\varepsilon$ describes the size of the sections of the fibers, which are homothetical to $S$, whereas the parameter $\varepsilon$ accounts for the local density of the distribution of the fibers in $\mathcal{O}$ through the function $n_\varepsilon$ defined by

$$\begin{aligned} & n_\varepsilon(x) := \sum_{z \in I_\varepsilon} (\sharp J^z_\varepsilon) \, 1_{Y^z_\varepsilon}(\hat{x}), \quad J^z_\varepsilon := \left\{j \in \mathbb{N}, \ \omega^j_\varepsilon \in Y^z_\varepsilon\right\}, \quad \hat{x} := (x_1, x_2), \\ & Y^z_\varepsilon := \varepsilon z + \varepsilon Y, \quad Y := [-1/2, 1/2)^2, \quad I_\varepsilon := \left\{z \in \mathbb{Z}^2, \quad Y^z_\varepsilon \subset \widehat{\mathcal{O}}\right\}, \end{aligned} \tag{1.4}$$

where $\sharp A$ denotes the cardinal of a set $A$. Given $x \in \mathcal{O}$, the scalar $n_\varepsilon(x)$ is the number of points of $\Omega_\varepsilon$ included in the cell $Y^z_\varepsilon$ such that $\hat{x} \in Y^z_\varepsilon$, if this cell exists at all. Therefore, $n_\varepsilon(x)$ is an approximation of the number of fibers included in the parallelepiped $Y^z_\varepsilon \times (0, L)$ containing $x$. The assumption

$$0 \leq n_\varepsilon(x) \leq N \quad \text{in } \mathcal{O}, \quad N \in \mathbb{N}, \quad n_\varepsilon \overset{\star}{\rightharpoonup} n \ \text{ weak star in } \ L^\infty(\mathcal{O}), \tag{1.5}$$

ensures that the fibers do not concentrate in some lower dimensional subset of $\mathcal{O}$. We also suppose that

$$\min_{j,j'\in J_\varepsilon, j\neq j'} |\omega_\varepsilon^i - \omega_\varepsilon^j| > R_\varepsilon, \qquad \operatorname{dist}(\Omega_\varepsilon, \partial\hat{\mathcal{O}}) > 5\sqrt{2}\varepsilon, \tag{1.6}$$

for some sequence of positive reals $(R_\varepsilon)$ satisfying (2.6). The hypothesis (1.6) guarantees that each fiber is separated by a sufficient distance from the other fibers and from the lateral boundary of $\mathcal{O}$. The periodic case corresponds to $\Omega = \{\varepsilon z,\ z\in I_\varepsilon\}$ and $n_\varepsilon$ given by $n_\varepsilon(x) = 1$ if $x \in \bigcup_{z\in I_\varepsilon} Y_\varepsilon^z \times (0,L)$, $n_\varepsilon(x) = 0$ otherwise. With no loss of generality, we assume that

$$0 \in \widehat{\mathcal{O}}, \qquad \overline{D} \subset S, \tag{1.7}$$

where $D$ denotes the open unit ball of $\mathbb{R}^2$. For simplicity, we suppose that (see Remark 2.1 (ii))

$$u_0 = 0, \qquad \text{if } \bar{k} = +\infty. \tag{1.8}$$

The density of electric energy is given in terms of two strictly convex functions $f$, $g$ satisfying a growth condition of order $p \in (1, +\infty)$ of the type

$$\alpha|\xi|^p \leq f(\xi), g(\xi) \leq \beta|\xi|^p \quad \forall \xi \in \mathbb{R}^3, \quad (\alpha, \beta > 0), \tag{1.9}$$

and is assumed to take large values in the fibers. More precisely, we suppose that

$$\lim_{\varepsilon\to 0} \lambda_\varepsilon \frac{\varepsilon^2}{r_\varepsilon^2 |S|} = \bar{k} \in (0, +\infty]. \tag{1.10}$$

# 2 Main result

We show that the effective behavior depends on the limit as $\varepsilon \to 0$ of the density of the capacities of the cross sections of the fibers with respect to the cross section of $\mathcal{O}$ in terms of some suitable notion of capacity defined by (2.5). This density proves to be of the order of magnitude of the parameter $\gamma^{(p)}$ given by

$$\gamma^{(p)} := \lim_{\varepsilon\to 0} \gamma_\varepsilon^{(p)}(r_\varepsilon) \in [0, +\infty], \quad \gamma_\varepsilon^{(p)}(r_\varepsilon) := \frac{r_\varepsilon^{2-p}}{\varepsilon^2} \quad \text{if } p \neq 2, \quad \gamma_\varepsilon^{(2)}(r_\varepsilon) := \frac{1}{\varepsilon^2 |\log r_\varepsilon|}. \tag{2.1}$$

A critical case occurs when $0 < \gamma^{(p)} < +\infty$. Then, a gap between the mean potential of the constituent parts of the composite may appear, giving rise to a concentration of electric energy stored in a thin region of space enveloping the fibers. The effective electric energy then takes the form of the sum of three terms like

$$\Phi(u, v) = \int_{\mathcal{O}} f(\nabla u)\ dx + \Phi_{cap}(v - u) + \Phi_{fibers}(v), \tag{2.2}$$

where $u$ stands for the weak limit in $W^{1,p}(\mathcal{O})$ of the sequence $(u_\varepsilon)$ of the solutions of (1.1), and $v$ represents a local approximation of the effective potential in the fibers. More precisely, the function $nv$, where $n$ is defined by (1.5), is the weak-$*$ limit in $\mathcal{M}_b(\overline{\mathcal{O}})$ of the sequence of measures $(u_\varepsilon \mu_\varepsilon)$, where, denoting by $\mathcal{L}^3_{\lfloor\mathcal{O}}$ the Lebesgue measure on $\mathcal{O}$,

$$\mu_\varepsilon := \frac{\varepsilon^2}{r_\varepsilon^2 |S|} \mathbb{1}_{T_{r_\varepsilon}}(x) \mathcal{L}^3_{\lfloor\mathcal{O}}. \tag{2.3}$$

The functional $\Phi_{fibers}$ accounts for the effective electric energy stored inside the fibers and is given by

$$\Phi_{fibers}(v) = \int_{\mathcal{O}} g^{hom}(\partial_3 v) n dx,$$

where $n$ is defined by (1.5) and $g^{hom}:\mathbb{R}\to\mathbb{R}$ is given by

$$g^{hom}(a) := \min\left\{g(q) : q\in\mathbb{R}^3,\ q_3 = a\right\}. \tag{2.4}$$

The functional $\Phi_{cap}$ describes the last mentioned concentration of energy in terms of the gap $v-u$ between the effective potential in the fibers and in the matrix. It takes the form

$$\Phi_{cap}(v-u) = \int_{\mathcal{O}} c^f(S; v-u)n dx.$$

The function $c^f(S;.)$ derives from the capacity $\mathrm{cap}^f$ defined, for all couple $(U,V)$ of open subsets of $\mathbb{R}^2$ such that $\overline{U}\subset V$ and for all $\alpha\in\mathbb{R}$, by

$$\mathrm{cap}^f(U,V;\alpha) = \inf\left\{\int_V f(\partial_1\varphi,\partial_2\varphi,0)\ d\widehat{x} : \varphi\in\mathcal{D}(V);\ \varphi=\alpha \text{ in } U\right\}. \tag{2.5}$$

To compute $c^f(S;.)$, we fix some sequence of positive reals $(R_\varepsilon)$ satisfying (see (2.1))

$$r_\varepsilon \ll R_\varepsilon \ll \varepsilon, \quad 1 << \gamma_\varepsilon^{(p)}(R_\varepsilon), \tag{2.6}$$

and study the asymptotic behavior of the sequence $(c_\varepsilon^f(R_\varepsilon))$ defined by

$$c_\varepsilon^f(R_\varepsilon) := \frac{1}{\varepsilon^2}\mathrm{cap}^f(r_\varepsilon S, R_\varepsilon D;\alpha). \tag{2.7}$$

This study reveals striking differences depending on the rate of growth $p$ of $f$ (see Section 6). Under assumption (2.1) we find that the sequence $(c_\varepsilon^f(R_\varepsilon))$ has an asymptotic behavior similar to that of the sequence $(\gamma_\varepsilon^{(p)}(r_\varepsilon))$. We also prove that $(c_\varepsilon^f(R_\varepsilon))$ is convergent in $\overline{\mathbb{R}}$ if $p\neq 2$ or if $p=2$ and $\gamma^{(2)}\in\{0,+\infty\}$. Actually we do not know whether it is convergent if $p=2$ and $0<\gamma^{(2)}<+\infty$. Therefore, in this case, we fix a converging subsequence $(c_{\varepsilon_k}^f(R_{\varepsilon_k}))$ and study $(\mathcal{P}_{\varepsilon_k})$. We show that then, for any sequence $(R'_\varepsilon)$ satisfying (2.6), the sequence $(c_{\varepsilon_k}^f(R'_{\varepsilon_k}))$ is also convergent. We obtain

$$\begin{aligned} c^f(S;\alpha) &:= \lim_{\varepsilon\to 0} c_\varepsilon^f(R_\varepsilon)\in[0,+\infty] && \text{if } p\neq 2 \text{ or if } p=2 \text{ and } \gamma^{(2)}\in\{0,+\infty\},\\ c^f(S;\alpha) &:= \lim_{k\to+\infty} c_{\varepsilon_k}^f(R_{\varepsilon_k})\in]0,+\infty[ && \text{if } p=2 \text{ and } 0<\gamma^{(2)}<+\infty. \end{aligned} \tag{2.8}$$

The extended real $c^f(S;\alpha)$ defined by (2.8) proves to be independent of the choice of the sequence $(R_\varepsilon)$ satisfying (2.6) (see (2.11) and Proposition 6.1 (ix)). The application $\alpha\to c^f(S;\alpha)$ turns out to be positively homogeneous of degree $p$, that is (under the convention $\infty.0=0$)

$$c^f(S;\alpha) = c^f(S;\mathrm{sgn}(\alpha))|\alpha|^p \quad \forall\alpha\in\mathbb{R}. \tag{2.9}$$

The extended reals $c^f(S,\pm 1)$ can be expressed in terms of $\gamma^{(p)}$ and of the "$p$-recession" function of $f$, that is the convex function, positively homogeneous of degree $p$, defined by

$$f^{\infty,p}(\xi) = \limsup_{t\to+\infty}\frac{f(t\xi)}{t^p}.$$

We assume that there exists $\alpha'>0$, $0<\beta'<p$ such that for all $\xi\in\mathbb{R}^3$

$$|f(\xi)-f^{\infty,p}(\xi)| \leq \alpha'(1+|\xi|^{\beta'}). \tag{2.10}$$

Under this hypothesis, we get (see Proposition 6.1 (viii))

$$\begin{aligned} c^f(S;\pm 1) &= \gamma^{(p)}\mathrm{cap}^{f^{\infty,p}}(S,\mathbb{R}^2;\pm 1) && \text{if} \quad p<2,\\ c^f(S;\pm 1) &= c^{f^{\infty,2}}(\pm 1)\gamma^{(2)} && \text{if} \quad p=2,\\ c^f(S;\pm 1) &= \gamma^{(p)} = +\infty && \text{if} \quad p>2. \end{aligned} \tag{2.11}$$

In the case $p=2$, we obtain no explicit formula giving the constant $c^{f^{\infty,2}}$ in terms of $\mathrm{cap}^f$ like for $p\neq 2$. This means that the real numbers $c^{f^{\infty,2}}(\pm 1)$ are simply defined by (2.8), (2.11). An explicit computation

is possible in some simple cases: for instance $c^{|\cdot|^2}(1) = c^{|\cdot|^2}(-1) = \pi$ (see (6.11)). A peculiar feature of the case $p = 2$ is that $c^{f^{\infty,2}}$ does not depend on $S$ (see Proposition 6.1 (ix)), whereas $c^f(S; \pm 1)$ does if $p < 2$ and $0 < \gamma^{(p)} < +\infty$.
The limiting problem associated with (1.1) is given by

$$(\mathcal{P}^{hom}): \quad \min \left\{ F^{hom}(u) - \int_{\mathcal{O}} q_b u \, dx - \int_{\Gamma_1} q_s u \, d\mathcal{H}^2 : u \in u_0 + W^{1,p}_{\Gamma_0}(\mathcal{O}) \right\}, \tag{2.12}$$

where (see (2.4), (2.11))

$$\begin{aligned} &F^{hom}(u) = \inf \left\{ \Phi(u,v) : v \in L^p(\mathcal{O}) \right\}, \\ &\Phi(u,v) = \begin{cases} \displaystyle\int_{\mathcal{O}} f(\nabla u) \, dx + \int_{\mathcal{O}} c^f(S; v-u) n dx + \bar{k} \int_{\mathcal{O}} g^{hom}(\partial_3 v) \, n dx & \\ \qquad\qquad\qquad \text{if } (u,v) \in \left(u_0 + W^{1,p}_{\Gamma_0}(\mathcal{O})\right) \times V_p, & \\ +\infty \qquad\qquad\qquad \text{otherwise,} & \end{cases} \\ &V_p := \left\{ v \in L^p(\mathcal{O}) : \partial_3 v \in L^p(\mathcal{O}), v = u_0 \text{ on } \Gamma_0 \cap \left(\widehat{\mathcal{O}} \times \{0, L\}\right) \right\}. \end{aligned} \tag{2.13}$$

The result stated in the next theorem in the case $p = 2$, $0 < \gamma^{(2)} < +\infty$ concerns the subsequence $(u_{\varepsilon_k})$. For notational simplicity, this subsequence is still denoted by $(u_\varepsilon)$.

**Theorem 2.1.** *Assume (1.2-1.10), (1.5), (2.1), (2.10), then the unique solution $u_\varepsilon$ of (1.1) converges weakly in $W^{1,p}(\mathcal{O})$ as $\varepsilon$ tends to 0 toward the unique solution $u$ to (2.12). Moreover, there holds*

$$\lim_{\varepsilon \to 0} \left\{ F_\varepsilon(u_\varepsilon) - \int_{\mathcal{O}} q_b u_\varepsilon \, dx - \int_{\Gamma_1} q_s u_\varepsilon \, d\mathcal{H}^2 \right\} = F^{hom}(u) - \int_{\mathcal{O}} q_b u \, dx - \int_{\Gamma_1} q_s u \, d\mathcal{H}^2. \tag{2.14}$$

*In addition, if $\gamma^{(p)} > 0$, then the sequence of measures $(u_\varepsilon \mu_\varepsilon)$, where $\mu_\varepsilon$ is defined by (2.3), weak $*$ converges in $\mathcal{M}_b(\overline{\mathcal{O}})$ to $nv\mathcal{L}^3_{\lfloor \mathcal{O}}$, where $n$ is defined by (1.5) and $v$ is the unique element of $V_p$, given by (2.13), such that $F^{hom}(u) = \Phi(u,v)$.*

**Remark 2.1.** *(i) If $\gamma^{(p)} = 0$, the variables $u$, $v$ are independent and the effective energy simply reads*

$$F^{hom}(u) = \int_{\mathcal{O}} f(\nabla u) \, dx + C, \quad C := \inf_{v \in V_p} \int_{\mathcal{O}} g^{hom}(\partial_3 v) dx \qquad (\gamma^{(p)} = 0).$$

*If $\gamma^{(p)} = +\infty$ (in particular when $p > 2$), the functional $\Phi(u,v)$ takes infinite values unless $u = v$, hence*

$$F^{hom}(u) = \int_{\mathcal{O}} f(\nabla u) \, dx + \bar{k} \int_{\mathcal{O}} g^{hom}(\partial_3 u) \, n dx \qquad (\gamma^{(p)} = +\infty),$$

*and the effective energy is that of the matrix augmented by a permittivity term in the direction of the fibers.*
*If $0 < \gamma^{(p)} < +\infty$, the effective electric energy is not a local functional. This means that it can not be written as the integration over $\mathcal{O}$ of a density of electric energy of the form $h(x, u(x), \nabla u(x), ...)$. By introducing the additional state variable $v$, we can write the effective energy under the form of a local functional of the couple $(u,v)$. This internal or hidden state variable is the limit of a suitable scaled of the electric potential in the sole fibers and accounts for the micro-structure. The total effective electric energy is that of a body totally filled up by the matrix material augmented by a term which is the infimal convolution of the last mentioned permittivity term supplied by the periodic distribution of fibers and a bonding term depending on the gap of electric potentials in the matrix and in the fibers. These concentrations of electric energy in the matrix in the immediate vicinity of the fibers, which may occur only when $p \leq 2$, induce a total effective energy lower than $\Phi(u,u)$. The structure of $\Phi$ stems from the contribution of each term entering the decomposition:*

$$F_\varepsilon(u) = \int_{\mathcal{O} \setminus (D_{R_\varepsilon} \times (0,L))} f(\nabla u) \, dx + \int_{(D_{R_\varepsilon} \times (0,L)) \setminus T_{r_\varepsilon}} f(\nabla u) \, dx + \lambda_\varepsilon \int_{T_{r_\varepsilon}} g(\nabla u) \, dx, \tag{2.15}$$

*where, given $(R_\varepsilon)$ satisfying (2.6), the set $D_{R_\varepsilon} \times (0,L)$ is the $R_\varepsilon$-neighborhood of the fibers defined by (4.3). The set $(D_{R_\varepsilon} \times (0,L)) \setminus T_{r_\varepsilon}$ is a small portion of the matrix surrounding the fibers where electric energy may concentrate due to the gap between the mean electric potentials in the fibers and in the matrix. This will provide a limit capacitary term associated with $f^{\infty,p}(\widehat{\nabla} u, 0)$ on $R_\varepsilon D \setminus r_\varepsilon S$. The contribution of $\mathcal{O} \setminus (D_{R_\varepsilon} \times (0,L))$ is obvious and the contribution of the fibers is classical (see [1, 18]).*

*(ii) The simplifying assumption (1.8) ensures that the effective electric energy stored in the fibers vanishes if $\bar{k} = +\infty$. An alternative is to assume that $u_0$ takes the same values on the intersection of the opposite bases of $\mathcal{O}$ with $\Gamma_0$.*

# 3 Conjecture for the case of a random distribution of fibers

In this section, we indicate a possible generalization of the periodic model to the case of parallel fibers randomly distributed in accordance with a stationary point process. In the model under consideration, the cross sections are not uniformly (i.e., periodically) distributed but their distribution is periodic in law i.e., the probability of presence of the sections is invariant under a suitable group $(\tau_z)_{z\in\mathbb{Z}^2}$ defined below. In the stochastic homogenization framework, the distribution of the sections is then said to be statistically homogeneous. We are going to give some precisions on this model.
Let us first define the discrete dynamical system $(\Omega, \boldsymbol{P}, (\tau_z)_{z\in Z^2})$ that models the distribution of the sections of the fibers. Given $d > 0$, we set

$$\Omega := \left\{(\omega_i)_{i\in\mathbb{N}} : \omega_i \in \mathbb{R}^2,\ |\omega_k - \omega_l| \geq d \text{ for } k \neq l\right\}, \tag{3.1}$$

and denote by $\Sigma$ the trace of the Borel $\sigma$-algebra of $(\mathbb{R}^2)^\infty$ on $\Omega$. We equip $\Omega$ with the group $(\tau_z)_{z\in\mathbb{Z}^2}$ defined by

$$\tau_z \omega = \omega - z,$$

where $\omega - z$ must be understood as $(\omega_i - z)_{i\in\mathbb{N}}$, and we denote by $\mathcal{F}$ the $\sigma$-algebra made up of all the events of $\Sigma$ which are invariant under the group $(\tau_z)_{z\in\mathbb{Z}^2}$. We assume the existence of a probability measure $\boldsymbol{P}$ on $(\Omega, \Sigma)$ for which $(\tau_z)_{z\in\mathbb{Z}^2}$ is a measure preserving transformation, i.e.,

$$\boldsymbol{P}\#\tau_z = \boldsymbol{P} \text{ for all } z \in \mathbb{Z}^2,$$

where $\boldsymbol{P}\#\tau_z$ denotes the pushforward of the probability measure $\boldsymbol{P}$ by the application $\tau_z$. For any measurable function $X : \Omega \to \mathbb{R}$, we denote by $\boldsymbol{E}^{\mathcal{F}} X$ its conditional expectation given $\mathcal{F}$, i.e., the unique $\mathcal{F}$-measurable function satisfying

$$\int_E \boldsymbol{E}^{\mathcal{F}} X \ d\boldsymbol{P} = \int_E X \ d\boldsymbol{P} \text{ for every } E \in \mathcal{F}.$$

Note that $\boldsymbol{E}^{\mathcal{F}} X$ is $\tau_z$-invariant (hence periodic) and that under the additional ergodic hypothesis which asserts that $\mathcal{F}$ is trivial, that is made up of events with probability measure 0 or 1, $\boldsymbol{E}^{\mathcal{F}} X$ is constant and nothing but the expectation $\boldsymbol{E}(X) := \int_\Omega X \ d\boldsymbol{P}$. Note also that the following asymptotic independance hypothesis

$$\lim_{|z|\to+\infty} \boldsymbol{P}(E \cap \tau_z E') = \boldsymbol{P}(E)\boldsymbol{P}(E'), \tag{3.2}$$

is a stronger but more intuitive condition yielding ergodicity.

The random set of fibers is defined by

$$T_{r_\varepsilon}(\omega) := \bigcup_{j\in J_\varepsilon(\omega)} T^j_{r_\varepsilon}, \quad T^j_{r_\varepsilon} := (\varepsilon\omega_j + r_\varepsilon S) \times (0,L), \quad J_\varepsilon(\omega) := \left\{j \in \mathbb{N}, \quad \omega_j \in \widehat{\mathcal{O}}\right\}. \tag{3.3}$$

We will denote by $(\mathcal{P}_\varepsilon(\omega))$ the problem associated with the random functional $F_\varepsilon(\omega, .)$.
Consider the random function

$$n_0 : \Omega \to \mathbb{N}, \quad \omega \mapsto n_0(\omega) := \#\left\{i \in \mathbb{N} :\ \omega_i \in \widehat{Y}\right\}, \quad \widehat{Y} := [0,1[^2. \tag{3.4}$$

In all likelyhood, the conditional expectation $\boldsymbol{E}^{\mathcal{F}} n_0(\omega)$ is the only additional corrector of the limit energy obtained in the periodic case. More precisely let us denote by $\Phi(\omega, .)$ the random functional:

$$\Phi(\omega,u,v)=\begin{cases}\displaystyle\int_{\mathcal{O}} f(\nabla u)\ dx+\boldsymbol{E}^{\mathcal{F}} n_0(\omega)\bar{k}\int_{\mathcal{O}} g^{hom}(\partial_3 v)\ dx+\boldsymbol{E}^{\mathcal{F}} n_0(\omega)\int_{\mathcal{O}} c^f(S;v-u)dx;\\ \qquad \text{if } (u,v)\in\big(u_0+W^{1,p}_{\Gamma_0}(\mathcal{O})\big)\times V_p,\\ +\infty \text{ otherwise,}\end{cases} \tag{3.5}$$

and set $F^{hom}(\omega,u)=\inf\{\Phi(\omega,u,v): v\in L^p(\mathcal{O}\}$. Then one may reasonably conjecture that

**Conjecture 3.1.** *Under assumptions stated above, when $\varepsilon$ tends to 0, the unique random solution $u_\varepsilon(\omega)$ to the problem $(\mathcal{P}_\varepsilon(\omega))$, deduced from (1.1) by substituting (3.3) for (1.3), almost surely weakly converges in $W^{1,p}(\mathcal{O})$ toward the unique solution $u(\omega)$ to*

$$(\overline{\mathcal{P}}(\omega))\quad \min\left\{F^{hom}(\omega,u)-\int_{\mathcal{O}} q_b u\ dx-\int_{\Gamma_1} q_s u\ d\mathcal{H}^2 : u\in u_0+W^{1,p}_{\Gamma_0}(\mathcal{O})\right\}.$$

*Moreover,*

$$\lim_{\varepsilon\to 0}\left\{F_\varepsilon(\omega,u_\varepsilon)-\int_{\mathcal{O}} q_b u_\varepsilon\ dx-\int_{\Gamma_1} q_s u_\varepsilon\ d\mathcal{H}^2\right\}=F^{hom}(\omega,u)-\int_{\mathcal{O}} q_b u\ dx-\int_{\Gamma_1} q_s u\ d\mathcal{H}^2,$$

*and, if $\gamma^{(p)}>0$, $v_\varepsilon(\omega):=\dfrac{\varepsilon^2}{r^2|D|}\mathbb{1}_{T_{r_\varepsilon}(\omega)}u_\varepsilon(\omega)$ almost surely weak $*$ converges in $\mathcal{M}_b(\overline{\mathcal{O}})$ to some $v(\omega)$ belonging to $V_p$ such that $F^{hom}(\omega,u)=\Phi(\omega,u(\omega),v(\omega))$. Furthermore, under the ergodic hypothesis (for instance under condition (3.2)), there holds $\boldsymbol{E}^{\mathcal{F}} n_0(\omega)=\boldsymbol{E} n_0$ so that the functionals $\Phi$, $F^{hom}$ and the functions $u$ and $v$ are deterministic.*

We hope to treat the mathematical analysis in a forthcoming paper.

# 4 Technical preliminaries and a priori estimates

The proof of Theorem 2.1 rests on an extensive investigation into the asymptotical behavior of the sequence $(u_\varepsilon)$ of the solutions to (1.1) and, more generally, of sequences $(u_\varepsilon)$ satisfying

$$\sup_{\varepsilon>0} F_\varepsilon(u_\varepsilon)<+\infty. \tag{4.1}$$

A commonly used method consists of introducing auxiliary sequences designed to characterize the comportment of the diverse constituents of the composite. The delicate step lies in the analysis of the fibers' behavior. An interesting approach consists in investigating the sequence $(u_\varepsilon\mu_\varepsilon)$, where $\mu_\varepsilon$ denotes the measure supported on the fibers defined by (2.3). To that aim, given a sequence $(R_\varepsilon)$ satisfying (2.6), we introduce the operators $\langle.\rangle_{R_\varepsilon}, \langle.\rangle_{r_\varepsilon}, \langle\langle.\rangle\rangle_\varepsilon$ defined on $L^p((0,L);W^{1,p}(\mathcal{O}))$ by setting

$$\begin{aligned}\langle\varphi\rangle_{R_\varepsilon}(x)&:=\sum_{j\in J_\varepsilon}\Big(\fint_{\partial D^j_{R_\varepsilon}}\varphi(\widehat{s},x_3)\ d\mathcal{H}^1(\widehat{s})\Big)\mathbb{1}_{D^j_{R_\varepsilon}}(\widehat{x}),\\ \langle\varphi\rangle_{r_\varepsilon}(x)&:=\sum_{j\in J_\varepsilon}\Big(\fint_{\partial D^j_{r_\varepsilon}}\varphi(\widehat{s},x_3)\ d\mathcal{H}^1(\widehat{s})\Big)\mathbb{1}_{D^j_{R_\varepsilon}}(\widehat{x}),\\ \langle\langle\varphi\rangle\rangle_\varepsilon&:=\sum_{z\in I_\varepsilon}\left(\fint_{Y^z_\varepsilon}\varphi(s,x_3)d\hat{s}\right)\mathbb{1}_{Y^z_\varepsilon}(\hat{x}),\end{aligned} \tag{4.2}$$

where

$$D^j_{R_\varepsilon}=\omega^j_\varepsilon+R_\varepsilon D,\qquad D_{R_\varepsilon}=\bigcup_{j\in J_\varepsilon} D^j_{R_\varepsilon}. \tag{4.3}$$

The series of estimates stated below will take a crucial part in the proof of Theorem 2.1 (the proof of Lemma 4.1 is situated at the end of Section 4).

**Lemma 4.1.** *There exists a constant $C$ such that for all $\varphi \in L^p((0,L); W^{1,p}(\widehat{\mathcal{O}}))$,*

$$
\begin{aligned}
&\int |\langle\varphi\rangle_{R_\varepsilon} - \langle\varphi\rangle_{r_\varepsilon}|^p \, d\mu_\varepsilon \leq \begin{cases} \frac{C}{\gamma_\varepsilon^{(p)}(r_\varepsilon)} \int_{\mathcal{O}} |\widehat{\nabla}\varphi|^p dx, & \text{if } p \leq 2, \\ \frac{C}{\gamma_\varepsilon^{(p)}(R_\varepsilon)} \int_{\mathcal{O}} |\widehat{\nabla}\varphi|^p dx, & \text{if } p > 2, \end{cases} \\
&\int |\langle\langle\varphi\rangle\rangle_\varepsilon - \langle\varphi\rangle_{r_\varepsilon}|^p \, d\mu_\varepsilon \leq \frac{C}{\gamma_\varepsilon^{(p)}(r_\varepsilon)} \int_{\mathcal{O}} |\widehat{\nabla}\varphi|^p dx, \\
&\int |\langle\langle\varphi\rangle\rangle_\varepsilon - \langle\varphi\rangle_{R_\varepsilon}|^p \, d\mu_\varepsilon \leq \frac{C}{\gamma_\varepsilon^{(p)}(R_\varepsilon)} \int_{\mathcal{O}} |\widehat{\nabla}\varphi|^p dx, \\
&\int_{Y_\varepsilon^z \times (0,L)} |\varphi - \langle\langle\varphi\rangle\rangle_\varepsilon|^p \, dx \leq C\varepsilon^p \int_{Y_\varepsilon^z \times (0,L)} |\widehat{\nabla}\varphi|^p dx \quad \forall z \in I_\varepsilon, \\
&\int |\varphi - \langle\varphi\rangle_{r_\varepsilon}|^p d\mu_\varepsilon \leq C r_\varepsilon^p \int |\widehat{\nabla}\varphi|^p d\mu_\varepsilon,
\end{aligned}
\tag{4.4}
$$

*where $\gamma_\varepsilon^{(p)}(.)$ is defined by (2.1).*

The next Lemma states a lower bound inequality for convex functionals on measures.

**Lemma 4.2.** *Let $\mathcal{O}$ be an open subset of $\mathbb{R}^N$ and let $\mu_\varepsilon$ and $\mu$ be bounded Radon measures in $\overline{\mathcal{O}}$ such that $\mu_\varepsilon$ weak $*$ converges in $\mathcal{M}_b(\overline{\mathcal{O}})$ toward $\mu$ and $f_\varepsilon$ a sequence of $\mu_\varepsilon$-measurable functions such that $\sup_\varepsilon \int_{\mathcal{O}} |f_\varepsilon|^p \, d\mu_\varepsilon < +\infty$. Then*

*i) the sequence of measures $(f_\varepsilon \mu_\varepsilon)$ is weak $*$ relatively compact in $\mathcal{M}_b(\overline{\mathcal{O}})$ and every cluster point $\nu$ is of the form $\nu = f\mu$ with $f \in L^p(\mathcal{O})$.*

*ii) If $f_\varepsilon \mu_\varepsilon \overset{\star}{\rightharpoonup} f\mu$, then $\liminf_{\varepsilon\to 0} \int j(f_\varepsilon) \, d\mu_\varepsilon \geq \int j(f) \, d\mu$ for all convex lower semi-continuous function $j$ on $\mathbb{R}$ satisfying a growth condition of order $p$. In addition*

$$
\begin{aligned}
\liminf_{\varepsilon\to 0} \int |f_\varepsilon^+|^p \, d\mu_\varepsilon &\geq \int |f^+|^p \, d\mu, \\
\liminf_{\varepsilon\to 0} \int |f_\varepsilon^-|^p \, d\mu_\varepsilon &\geq \int |f^-|^p \, d\mu.
\end{aligned}
\tag{4.5}
$$

*Proof.* The proof of this lemma is given in [5] with $j = \frac{1}{p}|\cdot|^p$ but the duality argument can be extended to any convex lower semi-continuous function satisfying a growth condition of order $p$. Assertion (4.5) results from the fact that if $f_\varepsilon^+ \mu_\varepsilon \overset{\star}{\rightharpoonup} g\mu$ and $f_\varepsilon \mu_\varepsilon \overset{\star}{\rightharpoonup} f\mu$, then $g \geq f^+ \; \mu$−*a.e.*, which can be easily checked by using positive continuous test functions (notice that in general, $g \neq f^+$). □

The main results of Section 4 are stated in the next Proposition, where the asymptotic behavior of several sequences associated to some sequence $(u_\varepsilon)$ satisfying (4.1) is specified.

**Proposition 4.1.** *Assume (1.9), (1.10), (2.1), (5.1). Let $(u_\varepsilon)$ be a sequence in $W^{1,p}(\mathcal{O})$ satisfying (4.1) and let $(\mu_\varepsilon)$, $(\langle u_\varepsilon\rangle_{R_\varepsilon})$ and $(\langle\langle u_\varepsilon\rangle\rangle_\varepsilon)$ be defined by (2.3), (4.2). Then the next estimates hold true*

$$
\begin{aligned}
&\int_{\mathcal{O}} |u_\varepsilon|^p + |\nabla u_\varepsilon|^p dx \leq C, \\
&\int |\partial_3 u_\varepsilon|^p + |u_\varepsilon|^p + |\langle u_\varepsilon\rangle_{r_\varepsilon}|^p + |\langle u_\varepsilon\rangle_{R_\varepsilon}|^p d\mu_\varepsilon \leq C,
\end{aligned}
\tag{4.6}
$$

*and there exists $u \in (u_0 + W^{1,p}_{\Gamma_0}(\mathcal{O}))$ and $v \in V_p$ such that, up to a subsequence, the next convergences take place*

$$
\begin{array}{llll}
u_\varepsilon \rightharpoonup u & & \textit{weakly in} & W^{1,p}(\mathcal{O}) \\
u_\varepsilon \mu_\varepsilon \overset{\star}{\rightharpoonup} n v \mathcal{L}^3_{\lfloor\mathcal{O}}, & \partial_3 u_\varepsilon \mu_\varepsilon \overset{\star}{\rightharpoonup} n \partial_3 v \mathcal{L}^3_{\lfloor\mathcal{O}} & \textit{weak} * \textit{in} & \mathcal{M}_b(\overline{\mathcal{O}}), \\
\langle u_\varepsilon\rangle_{R_\varepsilon} \mu_\varepsilon \overset{\star}{\rightharpoonup} n u \mathcal{L}^3_{\lfloor\mathcal{O}}, & \langle u_\varepsilon\rangle_{r_\varepsilon} \mu_\varepsilon \overset{\star}{\rightharpoonup} n v \mathcal{L}^3_{\lfloor\mathcal{O}} & \textit{weak} * \textit{in} & \mathcal{M}_b(\overline{\mathcal{O}}).
\end{array}
\tag{4.7}
$$

*In addition, $v = u$ if $\gamma^{(p)} = +\infty$ (in particular when $p > 2$).*

*Proof.* The first line of (4.6) follows from (4.1), the Dirichlet condition on $\Gamma_0$ and Poincaré inequality. We deduce that, up to a subsequence,

$$u_\varepsilon \rightharpoonup u \qquad \text{weakly in } W^{1,p}(\mathcal{O}), \tag{4.8}$$

for some $u \in W^{1,p}(\mathcal{O})$. We infer from the weak continuity of the trace application in $W^{1,p}(\mathcal{O})$ that $u \in (u_0 + W^{1,p}_{\Gamma_0}(\mathcal{O}))$. It follows from the fourth line of (4.4) that the sequence $(\langle\langle u_\varepsilon\rangle\rangle_\varepsilon)$ defined by (4.2) strongly converges to $u$ in $L^p(\mathcal{O})$. We deduce then from (1.5) that

$$\langle\langle u_\varepsilon\rangle\rangle_\varepsilon n_\varepsilon \rightharpoonup un \qquad \text{weakly in } L^p(\mathcal{O}). \tag{4.9}$$

We easily deduce from (2.3), (1.5), (4.2), and (4.9) that

$$\int |\langle\langle u_\varepsilon\rangle\rangle_\varepsilon|^p d\mu_\varepsilon \leq 9|n_\varepsilon|_{L^\infty(\mathcal{O})} \int_{\mathcal{O}} |\langle\langle u_\varepsilon\rangle\rangle_\varepsilon|^p dx \leq C. \tag{4.10}$$

On the other hand, by (1.5) and (2.3) we have

$$\mu_\varepsilon \overset{\star}{\rightharpoonup} n\mathcal{L}^3_{\lfloor\mathcal{O}} \quad \text{weak} * \text{in} \quad \mathcal{M}_b(\overline{\mathcal{O}}). \tag{4.11}$$

By applying Lemma 4.2, taking (4.10) and (4.11) into account, we deduce that there exists $f \in L^p(\mathcal{O})$ such that, up to a subsequence,

$$\langle\langle u_\varepsilon\rangle\rangle_\varepsilon \mu_\varepsilon \rightharpoonup fn\mathcal{L}^3_{\lfloor\mathcal{O}} \qquad \text{weak} * \text{in} \quad \mathcal{M}_b(\overline{\mathcal{O}}). \tag{4.12}$$

Testing the convergences (4.9) and (4.12) with a given $\varphi \in \mathcal{D}(\mathcal{O})$, taking the estimate $|\langle\langle\varphi\rangle\rangle_\varepsilon - \varphi| \leq C\varepsilon$ in $\mathcal{O}$ into account (this estimate is satisfied provided $\varepsilon$ is sufficiently small, see (1.4), (4.2)), we get

$$\lim_{\varepsilon\to 0}\int_{\mathcal{O}} \langle\langle\varphi\rangle\rangle_\varepsilon \langle\langle u_\varepsilon\rangle\rangle_\varepsilon n_\varepsilon dx = \int_{\mathcal{O}} \varphi un dx; \qquad \lim_{\varepsilon\to 0}\int \langle\langle\varphi\rangle\rangle_\varepsilon \langle\langle u_\varepsilon\rangle\rangle_\varepsilon d\mu_\varepsilon = \int_{\mathcal{O}} \varphi fn dx.$$

We prove below that

$$\lim_{\varepsilon\to 0}\left|\int_{\mathcal{O}} \langle\langle\varphi\rangle\rangle_\varepsilon \langle\langle u_\varepsilon\rangle\rangle_\varepsilon n_\varepsilon dx - \int \langle\langle\varphi\rangle\rangle_\varepsilon \langle\langle u_\varepsilon\rangle\rangle_\varepsilon d\mu_\varepsilon\right| = 0. \tag{4.13}$$

We deduce that $\int_{\mathcal{O}} \varphi un dx = \int_{\mathcal{O}} \varphi fn dx$ and then, by the arbitrary choice of $\varphi$, that $fn = un$ a.e. in $\mathcal{O}$. Therefore,

$$\langle\langle u_\varepsilon\rangle\rangle_\varepsilon \mu_\varepsilon \overset{\star}{\rightharpoonup} nu\mathcal{L}^3_{\lfloor\mathcal{O}} \quad \text{weak} * \text{in} \quad \mathcal{M}_b(\overline{\mathcal{O}}). \tag{4.14}$$

By (1.10) and (4.1), we have

$$\int |\partial_3 u_\varepsilon|^p \ d\mu_\varepsilon \leq C. \tag{4.15}$$

From (2.1), (4.4), (4.6), and (4.10), we derive

$$\int |u_\varepsilon|^p + |\langle u_\varepsilon\rangle_{r_\varepsilon}|^p + |\langle u_\varepsilon\rangle_{R_\varepsilon}|^p d\mu_\varepsilon \leq C,$$

which, joined with (4.15), yields (4.6). We deduce from (2.6), the third line of (4.4), and (4.14), that

$$\langle u_\varepsilon\rangle_{R_\varepsilon}\mu_\varepsilon \overset{\star}{\rightharpoonup} nu\mathcal{L}^3_{\lfloor\mathcal{O}} \quad \text{weak} * \text{in} \quad \mathcal{M}_b(\overline{\mathcal{O}}). \tag{4.16}$$

By (4.6) and Lemma 4.2,

$$\langle u_\varepsilon\rangle_{r_\varepsilon}\mu_\varepsilon \overset{\star}{\rightharpoonup} nv\mathcal{L}^3_{\lfloor\mathcal{O}}, \quad u_\varepsilon\mu_\varepsilon \overset{\star}{\rightharpoonup} nv_1\mathcal{L}^3_{\lfloor\mathcal{O}}, \quad \partial_3 u_\varepsilon\mu_\varepsilon \overset{\star}{\rightharpoonup} nw\mathcal{L}^3_{\lfloor\mathcal{O}} \quad \text{weak} * \text{in} \quad \mathcal{M}_b(\overline{\mathcal{O}}), \tag{4.17}$$

for some $(v, v_1, w) \in (L^p(\mathcal{O}))^3$. It follows from the estimate stated in the fifth line of (4.4) that

$$nv = nv_1 \text{ a.e. in } \mathcal{O}. \tag{4.18}$$

To show that

$$nw = n\partial_3 v \text{ a.e. in } \mathcal{O} \text{ and } nv = nu_0 \text{ on } \Gamma_0 \cap \widehat{\mathcal{O}} \times \{0, L\}), \tag{4.19}$$

it suffices (as in [5]) to pass to the limit in $\int \varphi \partial_3 u_\varepsilon \, d\mu_\varepsilon$ by integrating by parts with first $\varphi \in \mathcal{D}(\mathcal{O})$, next $\varphi$ of the form $\varphi(x) = \theta(\widehat{x})\psi(x_3)$ with $\theta \in \mathcal{D}(\mathcal{O}_0)$, $\mathcal{O}_0 = \{\widehat{x} \in \omega : (\widehat{x}, 0) \in \Gamma_0\}$, $\psi(0) = 1$, $\psi(L) = 0$ and finally $\theta \in \mathcal{D}(\mathcal{O}_L)$, $\mathcal{O}_L = \{\widehat{x} \in \omega : (\widehat{x}, L) \in \Gamma_0\}$, $\psi(0) = 1$, $\psi(L) = 0$.
Collecting (4.8), (4.16), (4.17), (4.18), (4.19), the convergences (4.7) are proved. It remains to notice that the first line of (4.4) yields $v = u$ when $p > 2$ or $\gamma^{(p)} = +\infty$: introducing an additional state variable to account for the asymptotic behavior of the electric potential in the fibers is not necessary!

**Proof of (4.13).** By (1.3), (1.4), (2.3), and (4.2) there holds

$$\left| \int_{\mathcal{O}} \langle\langle \varphi \rangle\rangle_\varepsilon \langle\langle u_\varepsilon \rangle\rangle_\varepsilon n_\varepsilon dx = \int \langle\langle \varphi \rangle\rangle_\varepsilon \langle\langle u_\varepsilon \rangle\rangle_\varepsilon d\mu_\varepsilon \right|, \qquad \text{if } T_{r_\varepsilon} \cap \bigcup_{z \in I_\varepsilon} \partial Y_\varepsilon^z \times (0, L) = \emptyset, \tag{4.20}$$

hence in this case there is nothing to prove. However, the equality (4.13) may fail to hold in the general case, because the border of some cells $Y_\varepsilon^z$ can possibly intersect some of the sections of the fibers. To circumvent this difficulty, we introduce the operator $\langle\langle . \rangle\rangle_{1,\varepsilon}$ defined by (see (1.4))

$$\begin{aligned} \langle\langle \varphi \rangle\rangle_{1,\varepsilon} &:= \sum_{z \in I_\varepsilon} \left( \fint_{Y_\varepsilon^z} \varphi(s, x_3) d\hat{s} \right) \mathbb{1}_{G_\varepsilon^z}(\hat{x}), \\ G_\varepsilon^z &:= \left( Y_\varepsilon^z \cup \bigcup_{j \in J_\varepsilon^z} S_{r_\varepsilon}^j \right) \setminus \bigcup_{j \in J_\varepsilon \setminus J_\varepsilon^z} S_{r_\varepsilon}^j. \end{aligned} \tag{4.21}$$

We deduce from (2.3), (4.2) and (4.21) that

$$\begin{aligned} \int \langle\langle \varphi \rangle\rangle_{1,\varepsilon} \langle\langle u_\varepsilon \rangle\rangle_{1,\varepsilon} d\mu_\varepsilon &= \sum_{j \in J_\varepsilon} \int_0^L \int_{S_{r_\varepsilon}^j} \langle\langle \varphi \rangle\rangle_{1,\varepsilon} \langle\langle u_\varepsilon \rangle\rangle_{1,\varepsilon} \frac{\varepsilon^2}{r_\varepsilon^2 |S|} dx \\ &= \sum_{z \in I_\varepsilon} \sum_{j \in J_\varepsilon^z} \int_0^L \int_{S_{r_\varepsilon}^j} \langle\langle \varphi \rangle\rangle_{1,\varepsilon} \langle\langle u_\varepsilon \rangle\rangle_{1,\varepsilon} \frac{\varepsilon^2}{r_\varepsilon^2 |S|} dx \\ &= \sum_{z \in I_\varepsilon} \sum_{j \in J_\varepsilon^z} \int_0^L \int_{S_{r_\varepsilon}^j} \left( \fint_{Y_\varepsilon^z} \varphi(s, x_3) d\hat{s} \right) \left( \fint_{Y_\varepsilon^z} u_\varepsilon(s, x_3) d\hat{s} \right) \frac{\varepsilon^2}{r_\varepsilon^2 |S|} dx \\ &= \sum_{z \in I_\varepsilon} \sum_{j \in J_\varepsilon^z} \int_0^L \int_{Y_\varepsilon^z} \left( \fint_{Y_\varepsilon^z} \varphi(s, x_3) d\hat{s} \right) u_\varepsilon(s, x_3) dx \\ &= \sum_{z \in I_\varepsilon} \int_0^L \int_{Y_\varepsilon^z} \left( \fint_{Y_\varepsilon^z} \varphi(s, x_3) d\hat{s} \right) \left( \fint_{Y_\varepsilon^z} u_\varepsilon(s, x_3) d\hat{s} \right) n_\varepsilon dx \\ &= \int_{\mathcal{O}} \langle\langle \varphi \rangle\rangle_\varepsilon \langle\langle u_\varepsilon \rangle\rangle_\varepsilon n_\varepsilon dx. \end{aligned} \tag{4.22}$$

By (4.22), the proof of (4.13) is achieved provided we establish that

$$\lim_{\varepsilon \to 0} \left| \int_{\mathcal{O}} \langle\langle \varphi \rangle\rangle_{1,\varepsilon} \langle\langle u_\varepsilon \rangle\rangle_{1,\varepsilon} - \langle\langle \varphi \rangle\rangle_\varepsilon \langle\langle u_\varepsilon \rangle\rangle_\varepsilon d\mu_\varepsilon \right| = 0. \tag{4.23}$$

To that aim, we notice that since $\varphi \in \mathcal{D}(\mathcal{O})$, by (4.2) and (4.21) the following estimate holds true:

$$|\langle\langle \varphi \rangle\rangle_{1,\varepsilon} - \langle\langle \varphi \rangle\rangle_\varepsilon| \leq C\varepsilon. \tag{4.24}$$

We deduce that

$$
\begin{aligned}
&\left|\int_{\mathcal{O}} \langle\langle\varphi\rangle\rangle_{1,\varepsilon}\langle\langle u_\varepsilon\rangle\rangle_{1,\varepsilon} - \langle\langle\varphi\rangle\rangle_{\varepsilon}\langle\langle u_\varepsilon\rangle\rangle_{\varepsilon} d\mu_\varepsilon\right| \\
&\qquad\qquad \le \int_{\mathcal{O}} |\langle\langle\varphi\rangle\rangle_{1,\varepsilon}||\langle\langle u_\varepsilon\rangle\rangle_{1,\varepsilon} - \langle\langle u_\varepsilon\rangle\rangle_{\varepsilon}|d\mu_\varepsilon + \int_{\mathcal{O}} |\langle\langle u_\varepsilon\rangle\rangle_{\varepsilon}||\langle\langle\varphi\rangle\rangle_{1,\varepsilon} - \langle\langle\varphi\rangle\rangle_{\varepsilon}|d\mu_\varepsilon \\
&\qquad\qquad \le C\int_{\mathcal{O}} |\langle\langle u_\varepsilon\rangle\rangle_{1,\varepsilon} - \langle\langle u_\varepsilon\rangle\rangle_{\varepsilon}|d\mu_\varepsilon + C\varepsilon\int_{\mathcal{O}} |\langle\langle u_\varepsilon\rangle\rangle_{\varepsilon}|d\mu_\varepsilon.
\end{aligned}
\tag{4.25}
$$

We prove below that

$$
\int_{\mathcal{O}} |\langle\langle u_\varepsilon\rangle\rangle_{1,\varepsilon} - \langle\langle u_\varepsilon\rangle\rangle_{\varepsilon}|d\mu_\varepsilon \le C\varepsilon, \tag{4.26}
$$

$$
\int_{\mathcal{O}} |\langle\langle u_\varepsilon\rangle\rangle_{\varepsilon}|d\mu_\varepsilon \le C. \tag{4.27}
$$

Assertion (4.23) results from (4.25), (4.26), and (4.27). Assertion (4.13) is proved. □

*Proof of (4.26).* By (1.4), (2.3), there holds

$$
\int_{\mathcal{O}} |\langle\langle u_\varepsilon\rangle\rangle_{1,\varepsilon} - \langle\langle u_\varepsilon\rangle\rangle_{\varepsilon}|d\mu_\varepsilon = \sum_{z\in I_\varepsilon}\sum_{j\in J^z_\varepsilon} \frac{\varepsilon^2}{r_\varepsilon^2|S|}\int_0^L\int_{S^j_{r_\varepsilon}} |\langle\langle u_\varepsilon\rangle\rangle_{1,\varepsilon} - \langle\langle u_\varepsilon\rangle\rangle_{\varepsilon}|dx. \tag{4.28}
$$

By (4.21), the function $\langle\langle u_\varepsilon\rangle\rangle_{1,\varepsilon}$ takes constant values on each set $S^j_{r_\varepsilon}\times\{x_3\}$, whereas the function $\langle\langle u_\varepsilon\rangle\rangle_{\varepsilon}$, defined by (4.2), may take up to four different values on $S^j_{r_\varepsilon}\times\{x_3\}$ if $S^j_{r_\varepsilon}\cap\partial Y^z_\varepsilon\neq\emptyset$ and $j\in J^z_\varepsilon$. For each $z\in I_\varepsilon$, we denote by $Z^z_\varepsilon$ the union of the cells $Y^{z'}_\varepsilon$ whose adherence has a non empty intersection with $\overline{Y^z_\varepsilon}$. The set $Z^z_\varepsilon$ is the subset of $\varepsilon(z+[-1,2[^2)$ defined by

$$
Z^z_\varepsilon := \bigcup_{k\in A_z} Y^{z+k}_\varepsilon, \quad A_z := (\mathbb{Z}^2\cap[-1,1]^2)\cap\left\{k\in\mathbb{Z}^2, z+k\in I_\varepsilon\right\}. \tag{4.29}
$$

Let us fix $z\in I_\varepsilon$. Noticing that $\#A_z\le 9$, we infer that for each $j\in J^z_\varepsilon$ and for a.e. $x_3\in(0,L)$, we have

$$
\begin{aligned}
&\frac{\varepsilon^2}{r_\varepsilon^2|S|}\int_{S^j_{r_\varepsilon}} |\langle\langle u_\varepsilon\rangle\rangle_{1,\varepsilon} - \langle\langle u_\varepsilon\rangle\rangle_{\varepsilon}|d\hat{x} \\
&\quad = \sum_{k\in A_z}\frac{\varepsilon^2}{r_\varepsilon^2|S|}\int_{S^j_{r_\varepsilon}\cap Y^{z+k}_\varepsilon}\left|\left(\fint_{Y^z_\varepsilon} u_\varepsilon(s,x_3)ds\right) - \left(\fint_{Y^{z+k}_\varepsilon} u_\varepsilon(s,x_3)ds\right)\right|d\hat{x} \\
&\quad \le \sum_{k\in A_z}\varepsilon^2\left|\left(\fint_{Y^z_\varepsilon} u_\varepsilon(s,x_3)ds\right) - \left(\fint_{Y^{z+k}_\varepsilon} u_\varepsilon(s,x_3)ds\right)\right| \\
&\quad \le C\sum_{k\in A_z}\int_{Z^z_\varepsilon}\left|\left(\fint_{Y^z_\varepsilon} u_\varepsilon(s,x_3)ds\right) - \left(\fint_{Y^{z+k}_\varepsilon} u_\varepsilon(s,x_3)ds\right)\right|d\hat{x} \\
&\quad \le C\sum_{k\in A_z}\int_{Z^z_\varepsilon}\left|u_\varepsilon(\hat{x},x_3) - \left(\fint_{Y^z_\varepsilon} u_\varepsilon(s,x_3)ds\right)\right| + \left|u_\varepsilon(\hat{x},x_3) - \left(\fint_{Y^{z+k}_\varepsilon} u_\varepsilon(s,x_3)ds\right)\right|d\hat{x} \\
&\quad \le C\sum_{k\in A_z}\int_{Z^z_\varepsilon}\left|u_\varepsilon(\hat{x},x_3) - \left(\fint_{Y^{z+k}_\varepsilon} u_\varepsilon(s,x_3)ds\right)\right|d\hat{x} \\
&\quad \le \sum_{k\in A_z} C_k\varepsilon\int_{Z^z_\varepsilon}\left|\widehat{\nabla}u_\varepsilon(\hat{x},x_3)\right|d\hat{x} \le C\varepsilon\int_{Z^z_\varepsilon}\left|\widehat{\nabla}u_\varepsilon(\hat{x},x_3)\right|d\hat{x}.
\end{aligned}
\tag{4.30}
$$

The next to last inequality in (4.30) being deduced from a change of variables in Poincaré-Wirtinger inequality $\int_{]-1,2[^2}\left|\varphi-\fint_{k+]0,1[^2}\varphi ds\right|dx\leq C_k\int_{]-1,2[^2}|\nabla\varphi|dx$ in $W^{1,1}(]-1,2[^2)$. Noticing that by (1.4) and (1.5) there holds $\sharp J_\varepsilon^z\leq N$, we deduce from (4.28), and (4.30) that

$$\begin{aligned}\int_{\mathcal{O}}|\langle\langle u_\varepsilon\rangle\rangle_{1,\varepsilon}-\langle\langle u_\varepsilon\rangle\rangle_\varepsilon|d\mu_\varepsilon&\leq C\varepsilon\sum_{z\in I_\varepsilon}\sharp J_\varepsilon^z\int_0^L\int_{Z_\varepsilon^z}\left|\widehat{\nabla}u_\varepsilon(\hat{x},x_3)\right|dx\\&\leq C\varepsilon\sum_{z\in I_\varepsilon}\int_0^L\int_{Z_\varepsilon^z}\left|\widehat{\nabla}u_\varepsilon(\hat{x},x_3)\right|dx.\end{aligned}\tag{4.31}$$

By (4.29), each set $Y_\varepsilon^z$ is included in at most 9 distinct sets $Z_\varepsilon^{z'}$, therefore by (4.6) we have

$$\begin{aligned}\varepsilon\sum_{z\in I_\varepsilon}\int_0^L\int_{Z_\varepsilon^z}\left|\widehat{\nabla}u_\varepsilon(\hat{x},x_3)\right|dx&\leq 9\varepsilon\sum_{z\in I_\varepsilon}\int_0^L\int_{Y_\varepsilon^z}\left|\widehat{\nabla}u_\varepsilon(\hat{x},x_3)\right|dx\\&\leq 9\varepsilon\int_{\mathcal{O}}\left|\widehat{\nabla}u_\varepsilon\right|dx\leq C\varepsilon\left(\int_{\mathcal{O}}\left|\widehat{\nabla}u_\varepsilon\right|^p dx\right)^{\frac{1}{p}}\leq C\varepsilon.\end{aligned}\tag{4.32}$$

The estimate (4.26) follows from (4.31) and (4.32). □

*Proof of (4.27).* By (2.3), (4.6), (4.2), and (4.29), there holds

$$\begin{aligned}\int_{\mathcal{O}}|\langle\langle u_\varepsilon\rangle\rangle_\varepsilon|d\mu_\varepsilon&=\sum_{z\in I_\varepsilon}\sum_{j\in J_\varepsilon^z}\frac{\varepsilon^2}{r_\varepsilon^2|S|}\int_0^L\int_{S_{r_\varepsilon}^j}|\langle\langle u_\varepsilon\rangle\rangle_\varepsilon|dx\\&=\sum_{z\in I_\varepsilon}\sum_{j\in J_\varepsilon^z}\sum_{k\in A_z}\frac{\varepsilon^2}{r_\varepsilon^2|S|}\int_0^L\int_{S_{r_\varepsilon}^j\cap Y_\varepsilon^{z+k}}\left|\fint_{Y_\varepsilon^{z+k}}u_\varepsilon(s,x_3)ds\right|dx\\&\leq C\sum_{z\in I_\varepsilon}\sum_{k\in A_z}\int_0^L\left|\int_{Y_\varepsilon^{z+k}}u_\varepsilon(s,x_3)ds\right|dx_3\leq C\int_{\mathcal{O}}|u_\varepsilon|dx\leq C,\end{aligned}\tag{4.33}$$

because $\sharp A_z$ and $\sharp J_\varepsilon^z$ are uniformely bounded. Assertion (4.27) is proved. □

**Proof of Lemma 4.1.** By (2.3) and (4.2), we have

$$\begin{aligned}\int|\langle\varphi\rangle_{R_\varepsilon}-\langle\varphi\rangle_{r_\varepsilon}|^p d\mu_\varepsilon&=\sum_{j\in J_\varepsilon}\frac{\varepsilon^2}{r_\varepsilon^2|S|}\int_0^L\int_{S_{r_\varepsilon}^j}|\langle\varphi\rangle_{R_\varepsilon}-\langle\varphi\rangle_{r_\varepsilon}|^p dx\\&=\sum_{j\in J_\varepsilon}\varepsilon^2\int_0^L\left|\langle\varphi\rangle_{R_\varepsilon}^j-\langle\varphi\rangle_{r_\varepsilon}^j\right|^p dx_3\\&=\sum_{j\in J_\varepsilon}\frac{\varepsilon^2}{R_\varepsilon^2|D|}\int_0^L\int_{D_{R_\varepsilon}^j}|\langle\varphi\rangle_{R_\varepsilon}-\langle\varphi\rangle_{r_\varepsilon}|^p dx,\end{aligned}\tag{4.34}$$

where $\langle\varphi\rangle_{r_\varepsilon}^j(x_3)$, $\langle\varphi\rangle_{R_\varepsilon}^j(x_3)$ denote the *constant* value taken by the functions $\langle\varphi\rangle_{r_\varepsilon}$ and $\langle\varphi\rangle_{R_\varepsilon}$ on $D_{R_\varepsilon}^j\times\{x_3\}$. The next inequality is proven in [5, Lemma A4] if $p\leq 2$ and is derived from the formula stated in [5, p. 433, l. -2] if $p>2$:

$$\begin{aligned}&\forall(R,\alpha)\in\mathbb{R}_+\times(0,1],\quad\int_{D_R}\left|\varphi-\fint_{\partial D_{\alpha R}}\varphi\,ds\right|^p dx\leq C\frac{R^p}{h(\alpha)}\int_{D_R}|\nabla\varphi|^p dx,\\&h(\alpha)=\alpha^{2-p}\text{ if }p\neq 2,\quad h(\alpha)=\frac{1}{1+|\log\alpha|}\text{ if }p=2.\end{aligned}\tag{4.35}$$

By (4.35) there holds, for a.e. $x_3 \in (0,L)$ and all $j \in J_\varepsilon$,

$$
\begin{aligned}
\int_{D^j_{R_\varepsilon}} |\langle\varphi\rangle_{R_\varepsilon} - \langle\varphi\rangle_{r_\varepsilon}|^p \, d\hat{x} &\leq C \int_{D^j_{R_\varepsilon}} |\varphi - \langle\varphi\rangle_{R_\varepsilon}|^p \, d\hat{x} + C \int_{D^j_{R_\varepsilon}} |\varphi - \langle\varphi\rangle_{r_\varepsilon}|^p \, d\hat{x} \\
&\leq \begin{cases} C \frac{R^p_\varepsilon}{h\left(\frac{r_\varepsilon}{R_\varepsilon}\right)} \int_{D^j_{R_\varepsilon}} |\nabla\varphi|^p \, d\hat{x}, & \text{if } p \leq 2, \\ C R^p_\varepsilon \int_{D^j_{R_\varepsilon}} |\nabla\varphi|^p \, d\hat{x}, & \text{if } p > 2. \end{cases}
\end{aligned} \tag{4.36}
$$

The first line of (4.4) follows from (2.1), (4.34), and (4.36). Similarly, setting

$$
J^z_\varepsilon := \left\{ j \in J_\varepsilon, \ \omega^j_\varepsilon \in Y^z_\varepsilon \right\}, \tag{4.37}
$$

and denoting by $\langle\langle\varphi\rangle\rangle^z_\varepsilon(x_3)$ the constant value taken by $\langle\langle\varphi\rangle\rangle_\varepsilon$ in $Y^z_\varepsilon \times \{x_3\}$, we get (see (4.2))

$$
\begin{aligned}
\int |\langle\langle\varphi\rangle\rangle_\varepsilon - \langle\varphi\rangle_{r_\varepsilon}|^p \, d\mu_\varepsilon &= \sum_{z\in I_\varepsilon} \sum_{j \in J^j_\varepsilon} \int_0^L \left|\langle\langle\varphi\rangle\rangle^z_\varepsilon - \langle\varphi\rangle^j_{r_\varepsilon}\right|^p \varepsilon^2 dx_3 \\
&= \sum_{z\in I_\varepsilon} \sum_{j \in J^j_\varepsilon} \int_0^L \left| \fint_{Y^z_\varepsilon} (\varphi - \langle\varphi\rangle^j_{r_\varepsilon})(\hat{x}, x_3) d\hat{x} \right|^p \varepsilon^2 dx_3 \\
&\leq C \sum_{z\in I_\varepsilon} \sum_{j \in J^j_\varepsilon} \int_0^L \int_{Y^z_\varepsilon} \left|\varphi - \langle\varphi\rangle^j_{r_\varepsilon}\right|^p dx.
\end{aligned} \tag{4.38}
$$

Noticing that by (1.6), (1.4) and (4.37) we have

$$
Y^z_\varepsilon \subset D(\omega^j_\varepsilon, \sqrt{2}\varepsilon) \subset Q^z_\varepsilon \subset \widehat{\mathcal{O}} \quad \forall z \in I_\varepsilon, \forall j \in J^z_\varepsilon, \qquad Q^z_\varepsilon := \varepsilon(z + 5Y), \tag{4.39}
$$

we infer from (4.2), (4.35) that for a.e. $x_3 \in (0,L)$, all $z \in I_\varepsilon$ and all $j \in J^z_\varepsilon$, there holds

$$
\begin{aligned}
\int_{Y^z_\varepsilon} \left|\varphi - \langle\varphi\rangle^j_{r_\varepsilon}\right|^p (s, x_3) d\hat{s} &\leq \int_{D(\omega^j_\varepsilon, \sqrt{2}\varepsilon)} \left|\varphi - \langle\varphi\rangle^j_{r_\varepsilon}\right|^p (s, x_3) d\hat{s} \\
&\leq C \frac{\varepsilon^p}{h\left(\frac{r_\varepsilon}{\varepsilon}\right)} \int_{D(\omega^j_\varepsilon, \sqrt{2}\varepsilon)} |\nabla\varphi|^p (s, x_3) d\hat{s} \\
&\leq C \frac{\varepsilon^p}{h\left(\frac{r_\varepsilon}{\varepsilon}\right)} \int_{Q^z_\varepsilon} |\nabla\varphi|^p (s, x_3) d\hat{s}.
\end{aligned} \tag{4.40}
$$

By (1.5) we have

$$
\sharp J^z_\varepsilon \leq N \quad \forall z \in I_\varepsilon. \tag{4.41}
$$

By (4.39) and (4.41), there holds

$$
\sum_{\{z \in I_\varepsilon, \ J^z_\varepsilon \neq \emptyset\}} \int_0^L \int_{Q^z_\varepsilon} |\nabla\varphi|^p \, dx \leq 25 \int_{\mathcal{O}} |\nabla\varphi|^p \, dx,
$$

we deduce from (4.38) and (4.40) that

$$
\int |\langle\langle\varphi\rangle\rangle_\varepsilon - \langle\varphi\rangle_{r_\varepsilon}|^p \, d\mu_\varepsilon \leq C \frac{\varepsilon^p}{h\left(\frac{r_\varepsilon}{\varepsilon}\right)} \sum_{z\in I_\varepsilon} \int_0^L \int_{Q^z_\varepsilon} |\nabla\varphi|^p \, dx \leq \frac{C}{\gamma^{(p)}_\varepsilon(r_\varepsilon)} \int_{\mathcal{O}} |\nabla\varphi|^p dx,
$$

hence the second line of (4.4) is proved. The third one is obtained in the same way and the fourth one is straightforward. The fifth one is easily derived by choosing $(R, \alpha) = (r_\varepsilon, 1)$ in (4.35). □

# 5 Proof of the main result

The demonstration of Theorem 2.1 is based on the Γ-convergence method (for precise details about this method, we refer the reader to [2], [3], [12]). The "lowerbound" and the "upperbound" stated respectively in Proposition 5.1 and Proposition 5.2, indicate in particular that the sequence of functionals $(F_\varepsilon)$ Γ-converges with respect to the strong topology of $L^p(\mathcal{O})$ to the functional $F^{hom}$ defined by (2.13). The proof of Theorem 2.1 is deduced in the following manner from the two last mentioned propositions and from Proposition 4.1:

## 5.1 Proof of Theorem 2.1.

We will only prove Theorem (2.1) in the most interesting case

$$\gamma^{(p)} > 0. \tag{5.1}$$

Let $(u_\varepsilon)$ be the sequence of the solutions to (1.1). By (1.8), and since $u_0$ is continous on $\overline{\mathcal{O}}$ (see (1.1)), there holds $F_\varepsilon(u_\varepsilon) - \int_{\mathcal{O}} q_B u_\varepsilon dx - \int_{\Gamma_1} q_s u_\varepsilon d\mathcal{H}^2 \leq F_\varepsilon(u_0) - \int_{\mathcal{O}} q_B u_0 dx - \int_{\Gamma_1} q_s u_0 d\mathcal{H}^2 \leq C$. As $\left| \int_{\mathcal{O}} q_B u_\varepsilon dx + \int_{\Gamma_1} q_s u_\varepsilon d\mathcal{H}^2 \right|^p \leq C F_\varepsilon(u_\varepsilon)$, we deduce that $(u_\varepsilon)$ satisfies (4.1). Therefore, we can apply Proposition 4.1 and, after possibly extracting a subsequence, assume that $(u_\varepsilon)$ converges weakly in $W^{1,p}(\mathcal{O})$ to some $u$, and that the sequence $(u_\varepsilon \mu_\varepsilon)$ weak $*$ converges in $\mathcal{M}_b(\overline{\mathcal{O}})$ to $v\mathcal{L}^3_{\lfloor \mathcal{O}}$ for some $v \in V_p$. We just have to prove that $(u, v)$ is the solution to (2.12). To that aim, we first apply Proposition 5.1, to get

$$\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) - \int_{\mathcal{O}} q_B u_\varepsilon dx - \int_{\Gamma_1} q_s u_\varepsilon d\mathcal{H}^2 \geq \Phi(u, v) - \int_{\mathcal{O}} q_B u dx - \int_{\Gamma_1} q_s u d\mathcal{H}^2. \tag{5.2}$$

By Proposition 5.2, there exists a sequence $(\varphi_\varepsilon)$ such that,

$$\varphi_\varepsilon \rightharpoonup u \quad \text{strongly in} \quad L^p(\mathcal{O}), \quad \varphi_\varepsilon \mu_\varepsilon \overset{\star}{\rightharpoonup} vn\mathcal{L}^3_{\lfloor \mathcal{O}} \quad \text{weak} * \text{ in } \mathcal{M}_b(\overline{\mathcal{O}}), \quad \limsup_{\varepsilon \to 0} F_\varepsilon(\varphi_\varepsilon) \leq \Phi(u, v). \tag{5.3}$$

Since $u_\varepsilon$ is the solution to (1.1), there holds

$$F_\varepsilon(u_\varepsilon) - \int_{\mathcal{O}} q_B u_\varepsilon dx - \int_{\Gamma_1} q_s u_\varepsilon d\mathcal{H}^2 \leq F_\varepsilon(\varphi_\varepsilon) - \int_{\mathcal{O}} q_B \varphi_\varepsilon dx - \int_{\Gamma_1} q_s \varphi_\varepsilon d\mathcal{H}^2. \tag{5.4}$$

We infer fom (5.2), (5.3), (5.4) and from the weak continuity on $W^{1,p}(\mathcal{O})$ of the linear form $\varphi \to \int_{\mathcal{O}} q_B \varphi dx - \int_{\Gamma_1} q_s \varphi d\mathcal{H}^2$ that

$$\Phi(u, v) - \int_{\mathcal{O}} q_B u dx - \int_{\Gamma_1} q_s u d\mathcal{H}^2 \leq \min(\mathcal{P}^{hom}),$$

hence $(u, v)$ is the unique solution to $(\mathcal{P}^{hom})$ (the uniqueness results from the strict convexity of $f$ and $g$). □

## 5.2 Lower bound

The result stated in the next proposition in the case $p = 2$, $0 < \gamma^{(2)} < +\infty$ concern only the subsequences $(u_{\varepsilon_k}), (F_{\varepsilon_k}), ...$ corresponding to the assumption (2.8). However, for notational simplicity, such subsequences will still denoted by $(u_\varepsilon), (F_\varepsilon), ....$

**Proposition 5.1.** *Under the assumptions of Theorem 2.1, for all $(u, v) \in (u_0 + W^{1,p}_{\Gamma_0}(\mathcal{O})) \times V_p$ and for all sequence $(u_\varepsilon)$ in $u_0 + W^{1,p}_{\Gamma_0}(\mathcal{O})$ which weakly converges in $W^{1,p}(\mathcal{O})$ toward $u$ and such that $(u_\varepsilon \mu_\varepsilon)$ weak $*$ converges in $\mathcal{M}_b(\overline{\mathcal{O}})$ to $vn\mathcal{L}^3_{\lfloor \mathcal{O}}$, we have*

$$\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \geq \Phi(u, v). \tag{5.5}$$

*Proof.* We can suppose that $\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) < +\infty$, otherwise there is nothing to prove. Accordingly, after possibly extracting a subsequence, we can assume that (4.1) is verified and that the estimates (4.6) and the convergences (4.7) established in Proposition 4.1 take place. We choose a suitable sequence $(R_\varepsilon)$ of positive reals satisfying (2.6) (the choice of $(R_\varepsilon)$ will be made more precise in Lemma 7.2), and establish (see below) that

$$\liminf_{\varepsilon\to 0} \int_{\mathcal{O}\setminus(D_{R_\varepsilon}\times(0,L))} f(\nabla u_\varepsilon)\,dx \geq \int_{\mathcal{O}} f(\nabla u)\,dx, \tag{5.6}$$

$$\liminf_{\varepsilon\to 0} \lambda_\varepsilon \int_{T_{r_\varepsilon}} g(\nabla u_\varepsilon)\,dx \geq \bar{k} \int_{\mathcal{O}} g^{hom}(\partial_3 v)\,ndx, \tag{5.7}$$

$$\liminf_{\varepsilon\to 0} \int_{(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}} f(\nabla u_\varepsilon)\,dx \geq \int_{\mathcal{O}} c^f(S; v-u)\,ndx \quad \text{otherwise}, \tag{5.8}$$

where $g^{hom}$ and $c^f$ are defined by (2.4) and (2.8). Collecting (5.6), (5.7), (5.8) we obtain (5.5) which, joined with (4.7), achieves the proof of Proposition 5.1. □

**Proof of (5.6).** By (2.6) and (4.3) we have $|D_{R_\varepsilon}\times(0,L)| \to 0$, therefore the sequence $\left(\mathbb{1}_{\mathcal{O}\setminus(D_{R_\varepsilon}\times(0,L))}\nabla u_\varepsilon\right)$ weakly converges in $L^p(\mathcal{O};\mathbb{R}^3)$ toward $\nabla u$. Assertion (5.6) then follows from the lower semi-continuity of $q \mapsto \int_{\mathcal{O}} f(q)\,dx$. □

**Proof of (5.7).** If $\bar{k} < +\infty$, by (1.10), (2.4), (4.7) and Lemma 4.2, we have:

$$\begin{aligned}\liminf_{\varepsilon\to 0} \lambda_\varepsilon \int_{T_{r_\varepsilon}} g(\nabla u_\varepsilon)\,dx &\geq \liminf_{\varepsilon\to 0} \lambda_\varepsilon \frac{r_\varepsilon^2|S|}{\varepsilon^2} \int g^{hom}(\partial_3 u_\varepsilon)\,d\mu_\varepsilon \\ &\geq \bar{k} \int_{\mathcal{O}} g^{hom}(\partial_3 v)\,ndx.\end{aligned}$$

Otherwise, if $\bar{k} = +\infty$, it is enough to notice that $\lambda_\varepsilon \int_{T_{r_\varepsilon}} g(\nabla u_\varepsilon)\,dx$ is bounded from below by 0. □

**Proof of (5.8).** If $\gamma^{(p)} = +\infty$ (in particular if $p > 2$), there is nothing to prove because then, by Proposition 4.1, $v = u$. From now on, we assume that $0 < \gamma^{(p)} < +\infty$ (hence $p \leq 2$). First, we show (see Lemma 7.1) that there exists an approximation $(\widehat{u}_\varepsilon)$ of $u_\varepsilon$ piecewise constant in $x_3$ satisfying

$$\begin{aligned}
&\lambda_\varepsilon \int_{T_{r_\varepsilon}} |\widehat{\nabla}\widehat{u}_\varepsilon|^p\,dx \leq \lambda_\varepsilon \int_{T_{r_\varepsilon}} |\nabla u_\varepsilon|^p\,dx, \quad \int_{\mathcal{O}} |\widehat{\nabla}\widehat{u}_\varepsilon|^p\,dx \leq \int_{\mathcal{O}} |\nabla u_\varepsilon|^p\,dx, \quad \widehat{\nabla}\widehat{u}_\varepsilon := (\partial_1\widehat{u}_\varepsilon, \partial_2\widehat{u}_\varepsilon, 0) \\
&\int |\langle\widehat{u}_\varepsilon\rangle_{r_\varepsilon}|^p + |\langle\widehat{u}_\varepsilon\rangle_{R_\varepsilon}|^p d\mu_\varepsilon \leq C, \\
&\langle\widehat{u}_\varepsilon\rangle_{r_\varepsilon}\mu_\varepsilon \stackrel{\star}{\rightharpoonup} nv\mathcal{L}^3_{\lfloor\mathcal{O}}, \qquad \langle\widehat{u}_\varepsilon\rangle_{R_\varepsilon}\mu_\varepsilon \stackrel{\star}{\rightharpoonup} nu\mathcal{L}^3_{\lfloor\mathcal{O}} \quad \text{weak} * \text{in} \quad \mathcal{M}_b(\overline{\mathcal{O}}), \\
&\liminf_{\varepsilon\to 0} \int_{(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}} f(\nabla u_\varepsilon)\,dx \geq \liminf_{\varepsilon\to 0} \int_{(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}} f^{\infty,p}(\widehat{\nabla}\widehat{u}_\varepsilon, 0)\,dx.
\end{aligned} \tag{5.9}$$

Next, we fix a positive real $\delta$ satisfying

$$1 < \delta < 2, \tag{5.10}$$

and define the set $S_{r_\varepsilon}^{-r_\varepsilon^\delta}$ by setting $(U,\alpha) = (S_{r_\varepsilon}, r_\varepsilon^\delta)$ in

$$\begin{aligned} U^{-\alpha} &:= \{\hat{x} \in U, \quad \text{dist}\,(x, \partial U) > \alpha\}, \\ U^{+\alpha} &:= \left\{\hat{x} \in \mathbb{R}^2, \quad \text{dist}\,(x, \partial U) < \alpha\right\} \cup U. \end{aligned} \tag{5.11}$$

Notice that by (1.7) we have, for small $\varepsilon$ small enough (see (4.3)),

$$D_{r_\varepsilon}^j \subset S_{r_\varepsilon}^{j,-r_\varepsilon^\delta} \quad \forall\ j \in J_\varepsilon. \tag{5.12}$$

We prove (see Lemma 7.3) that for a suitable choice of the sequence $(R_\varepsilon)$ satisfying (2.6) (the choice of this sequence is determined by Lemma 7.2), there exists an approximation $\widehat{u}_\varepsilon$ of $\widehat{u}_\varepsilon$ verifying

$$\begin{gathered}\int_{(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}} f^{\infty,p}(\widehat{\nabla}\widehat{u}_\varepsilon,0)\ dx \geq \int_{(D_{R_\varepsilon}\times(0,L))\setminus(S_{r_\varepsilon}^{-r_\varepsilon^\delta}\times(0,L)} f^{\infty,p}(\widehat{\nabla}\widehat{u}_\varepsilon,0)\ dx + o(1),\\ \widehat{u}_\varepsilon = \langle\widehat{u}_\varepsilon\rangle_{r_\varepsilon} = \langle\widehat{u}_\varepsilon\rangle_{r_\varepsilon} \quad \text{on} \quad \partial S_{r_\varepsilon}^{-r_\varepsilon^\delta}\times(0,L), \qquad \widehat{u}_\varepsilon = \langle\widehat{u}_\varepsilon\rangle_{R_\varepsilon} = \langle\widehat{u}_\varepsilon\rangle_{R_\varepsilon} \quad \text{on} \quad \partial D_{R_\varepsilon}\times(0,L),\end{gathered} \tag{5.13}$$

The properties of $\widehat{u}_\varepsilon$ allow us to make good use of the capacitary problem (2.5). More precisely, let us denote by $\langle\widehat{u}_\varepsilon\rangle^j_{r_\varepsilon}(x_3)$ (resp. $\langle\widehat{u}_\varepsilon\rangle^j_{R_\varepsilon}(x_3)$) the *constant* value taken by the function $\langle\widehat{u}_\varepsilon\rangle_{r_\varepsilon}$ (resp. $\langle\widehat{u}_\varepsilon\rangle_{R_\varepsilon}$) on the set $D^j_{R_\varepsilon}\times\{x_3\}$ (see (4.2)). By (5.13), for each $(j,x_3)\in J_\varepsilon\times(0,L)$, the function $\widehat{u}_\varepsilon$ is equal to $\langle\widehat{u}_\varepsilon\rangle^j_{R_\varepsilon}(x_3)$ on the set $\partial D^j_{R_\varepsilon}\times\{x_3\}$ and to $\langle\widehat{u}_\varepsilon\rangle^j_{r_\varepsilon}(x_3)$ $\partial S^{j,-r_\varepsilon^\delta}_{r_\varepsilon}\times\{x_3\}$ (see (5.11)). Therefore there holds, for all $j\in J_\varepsilon$ and for a.e. $x_3\in(0,L)$ (see (2.5))

$$\int_{D^j_{R_\varepsilon}\setminus S^{j,-r_\varepsilon^\delta}_{r_\varepsilon}} f^{\infty,p}(\widehat{\nabla}\widehat{u}_\varepsilon,0)(x)\ d\widehat{x} \geq \operatorname{cap}^{f^{\infty,p}}\big(S^{j,-r_\varepsilon^\delta}_{r_\varepsilon}, D^j_{R_\varepsilon}; \langle\widehat{u}_\varepsilon\rangle^j_{r_\varepsilon}(x_3) - \langle\widehat{u}_\varepsilon\rangle^j_{R_\varepsilon}(x_3)\big).$$

We deduce that

$$\begin{aligned}\int_{(D_{R_\varepsilon}\times(0,L))\setminus S_{r_\varepsilon}^{-r_\varepsilon^\delta}\times(0,L)} f^{\infty,p}(\widehat{\nabla}\widehat{u}_\varepsilon,0)\ dx &\geq \int_0^L \Big(\sum_{j\in J_\varepsilon}\int_{D^j_{R_\varepsilon}\setminus S^{j,-r_\varepsilon^\delta}_{r_\varepsilon}} f^{\infty,p}(\widehat{\nabla}\widehat{u}_\varepsilon,0)\ d\widehat{x}\Big)\ dx_3\\ &\geq \int_0^L \Big(\sum_{j\in J_\varepsilon}\operatorname{cap}^{f^{\infty,p}}\big(S^{j,-r_\varepsilon^\delta}_{r_\varepsilon}, D^j_{R_\varepsilon}; \langle\widehat{u}_\varepsilon\rangle^j_{r_\varepsilon} - \langle\widehat{u}_\varepsilon\rangle^j_{R_\varepsilon}\big)\Big)\ dx_3.\end{aligned} \tag{5.14}$$

Because $f^{\infty,p}$ is $p$ positively homogeneous, we can apply (6.10) and, for each $(j,x_3)\in J_\varepsilon\times(0,L)$, obtain (see (2.5), (5.11))

$$\begin{aligned}\operatorname{cap}^{f^{\infty,p}}\big(S^{j,-r_\varepsilon^\delta}_{r_\varepsilon}, D^j_{R_\varepsilon}; \langle\widehat{u}_\varepsilon\rangle^j_{r_\varepsilon} - \langle\widehat{u}_\varepsilon\rangle^j_{R_\varepsilon}\big) &= r_\varepsilon^{2-p}\operatorname{cap}^{f^{\infty,p}}\big(S^{-r_\varepsilon^{\delta-1}}, (R_\varepsilon/r_\varepsilon)D; \langle\widehat{u}_\varepsilon\rangle^j_{r_\varepsilon} - \langle\widehat{u}_\varepsilon\rangle^j_{R_\varepsilon}\big)\\ &= \frac{r_\varepsilon^{2-p}}{\varepsilon^2}\frac{\varepsilon^2}{r_\varepsilon^2|S|}\int_{S^j_{r_\varepsilon}}\operatorname{cap}^{f^{\infty,p}}\big(S^{-r_\varepsilon^{\delta-1}}, (R_\varepsilon/r_\varepsilon)D; \langle\widehat{u}_\varepsilon\rangle^j_{r_\varepsilon} - \langle\widehat{u}_\varepsilon\rangle^j_{R_\varepsilon}\big)d\hat{x}.\end{aligned} \tag{5.15}$$

Let us fix a bounded Lipschitz domain $S'$ such that

$$\overline{S}' \subset S. \tag{5.16}$$

For small $\varepsilon$'s, there holds $\overline{S}'\subset S^{-r_\varepsilon^{\delta-1}}$, therefore by (2.3), (5.14), (5.15), and (6.8), we have

$$\int_{(D_{R_\varepsilon}\times(0,L))\setminus S_{r_\varepsilon}^{-r_\varepsilon^\delta}\times(0,L)} f^{\infty,p}(\widehat{\nabla}\widehat{u}_\varepsilon,0)\ dx \geq \frac{r_\varepsilon^{2-p}}{\varepsilon^2}\int \operatorname{cap}^{f^{\infty,p}}\big(S', (R_\varepsilon/r_\varepsilon)D; \langle\widehat{u}_\varepsilon\rangle_{r_\varepsilon} - \langle\widehat{u}_\varepsilon\rangle_{R_\varepsilon}\big)d\mu_\varepsilon. \tag{5.17}$$

We then distinguish two cases.
**Case $p<2$.** Collecting (2.1), (5.9), (5.13), (5.17), and (6.5), we deduce that

$$\liminf_{\varepsilon\to0}\int_{(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}} f(\nabla u_\varepsilon)\ dx \geq \gamma^{(p)}\liminf_{\varepsilon\to0}\int_{\mathcal{O}}\operatorname{cap}^{f^{\infty,p}}\big(S',\mathbb{R}^2; \langle\widehat{u}_\varepsilon\rangle_{r_\varepsilon} - \langle\widehat{u}_\varepsilon\rangle_{R_\varepsilon}\big)\ d\mu_\varepsilon.$$

By applying Lemma 4.2 (ii) to the convex function $j(.) = \operatorname{cap}^{f^{\infty,p}}\big(S,\mathbb{R}^2;\cdot\big)$ which, for $p<2$, has a growth of order $p$ (see Proposition 6.1 (i) and (6.10), (6.12)), taking (4.7) and (4.11) into account, we infer

$$\liminf_{\varepsilon\to0}\int_{(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}} f(\nabla u_\varepsilon)\ dx \geq \gamma^{(p)}\int_{\mathcal{O}}\operatorname{cap}^{f^{\infty,p}}\big(S',\mathbb{R}^2; v-u\big)\ n dx, \tag{5.18}$$

for all Lipschitz domain $S'$ satisfying (5.16). Fixing an increasing sequence $(S_n)_{n\in\mathbb{N}}$ of Lipschitz domains such that $\overline{S}_n \subset S$ and $\bigcup_{n\in\mathbb{N}} S_n = S$, substituting $S_n$ for $S'$ in (5.18) and passing to the limit as $n \to +\infty$, thanks to (6.8), (6.9) and to the Monotone Convergence Theorem, we get (5.8).

**Case** $p = 2$**.** In this case, we fix two positive reals $r, R$ such that

$$r\overline{D} \subset S \subset RD, \tag{5.19}$$

and specify the choice of $S'$ by setting

$$S' := rD. \tag{5.20}$$

By (5.19), (7.13), (6.5), (6.8), and (6.10), we have

$$\begin{aligned} \operatorname{cap}^{f^{\infty,2}}(r_\varepsilon S', R_\varepsilon D; \pm 1) &= \operatorname{cap}^{f^{\infty,2}}(r_\varepsilon rD, R_\varepsilon D; \pm 1) = \operatorname{cap}^{f^{\infty,2}}\left(r_\varepsilon RD, R_\varepsilon \frac{R}{r} D; \pm 1\right) \\ &\geq \operatorname{cap}^{f^{\infty,2}}(r_\varepsilon S, R'_\varepsilon D; \pm 1). \end{aligned} \tag{5.21}$$

We deduce from (2.7), (2.8), (5.21), and (6.13) that

$$\liminf_{\varepsilon\to 0} \frac{\operatorname{cap}^{f^{\infty,2}}(r_\varepsilon S', R_\varepsilon D; \pm 1)}{\varepsilon^2} \geq \gamma^{(2)} c^{f^{\infty,2}}(\pm 1). \tag{5.22}$$

By (5.17) and (6.10), there holds

$$\begin{aligned} \int_{(D_{R_\varepsilon}\times(0,L))\setminus S^{-r^\delta_\varepsilon}_{r_\varepsilon}\times(0,L)} f^{\infty,p}(\widehat{\nabla}\widehat{u}_\varepsilon, 0)\, dx = & \frac{\operatorname{cap}^{f^{\infty,2}}\big(r_\varepsilon S', R_\varepsilon D; 1\big)}{\varepsilon^2} \int |(\langle \widehat{u}_\varepsilon\rangle_{r_\varepsilon} - \langle \widehat{u}_\varepsilon\rangle_{R_\varepsilon})^+|^2 d\mu_\varepsilon \\ & + \frac{\operatorname{cap}^{f^{\infty,2}}\big(r_\varepsilon S', R_\varepsilon D; -1\big)}{\varepsilon^2} \int |(\langle \widehat{u}_\varepsilon\rangle_{r_\varepsilon} - \langle \widehat{u}_\varepsilon\rangle_{R_\varepsilon})^-|^2 d\mu_\varepsilon. \end{aligned} \tag{5.23}$$

Joining Lemma 4.2 (see (4.5)), (5.13), (5.17), (5.22), (5.23), and (6.5), we get (5.8). □

## 5.3 Upper bound

As above, the result stated below in the case $p = 2$, $0 < \gamma^{(2)} < +\infty$ are obtained for subsequences $(a_{\varepsilon_k})$ corresponding to the assumption (2.8) which are still denoted by $(a_\varepsilon)$.

**Proposition 5.2.** *Under the assumptions of Theorem 2.1, for all* $(u, v) \in W^{1,p}_{\Gamma_0}(\mathcal{O}) \times V_p$*, there exists a sequence* $(u_\varepsilon)$ *such that*

$$\begin{aligned} & u_\varepsilon \rightharpoonup u \quad \textit{strongly in } L^p(\mathcal{O}), \quad u_\varepsilon\mu_\varepsilon \overset{\star}{\rightharpoonup} vn\mathcal{L}^3_{\lfloor\mathcal{O}} \quad \textit{weak} * \textit{in} \ \ \mathcal{M}_b(\overline{\mathcal{O}}), \\ & \limsup_{\varepsilon\to 0} \ F_\varepsilon(u_\varepsilon) \leq \Phi(u, v). \end{aligned} \tag{5.24}$$

*Proof.* By density and diagonalization arguments, (see [5, pp. 424–429] for more details), we are reduced to prove that for all $(u, v) \in (C^1(\overline{\mathcal{O}}))^2$ such that

$$\Phi(u, v) < +\infty, \tag{5.25}$$

there exists a sequence $(u_\varepsilon)$ in $W^{1,p}(\mathcal{O})$ (thanks to the truncature argument employed in [5, p. 428], we can forget the boundary constraint on $\Gamma_0$) such that

$$\begin{aligned} & u_\varepsilon \rightharpoonup u \ \ \text{strongly in } L^p(\mathcal{O}), \quad u_\varepsilon\mu_\varepsilon \overset{\star}{\rightharpoonup} vn\mathcal{L}^3_{\lfloor\mathcal{O}} \ \ \text{weak} * \text{in} \ \ \mathcal{M}_b(\overline{\mathcal{O}}), \\ & \limsup_{\varepsilon\to 0} \int_{\mathcal{O}\setminus T_{r_\varepsilon}} f(\nabla u_\varepsilon) dx + \lambda_\varepsilon \int_{T_{r_\varepsilon}} g(\nabla u_\varepsilon) dx \leq \Phi(u, v). \end{aligned} \tag{5.26}$$

Accordingly, let us fix $(u,v) \in (C^1(\overline{\mathcal{O}}))^2$ satisfying (5.25). By (1.9), (2.4) and the strict convexity of $g$, there exists a unique field $\varphi \in C(\overline{\mathcal{O}};\mathbb{R}^2)$ such that

$$g(\varphi_1(x),\varphi_2(x),\partial_3 v(x)) = g^{hom}(\partial_3 v(x)) \quad \forall x \in \overline{\mathcal{O}}. \tag{5.27}$$

If $p \neq 2$, we fix *any* sequence $(R_\varepsilon)$ satisfying (2.6). If $p = 2$, we set

$$R_\varepsilon := R'_\varepsilon. \tag{5.28}$$

We denote by $\theta_\varepsilon : \widehat{\mathcal{O}} \to \mathbb{R}$ the unique solution to the problem

$$\min\left\{\int_{\widehat{\mathcal{O}}} f^{\infty,p}(\widehat{\nabla}\theta(\hat{x}),0)\; d\widehat{x}, \quad \theta \in W^{1,p}(\widehat{\mathcal{O}}),\ \theta = 1 \text{ in } S_{r_\varepsilon}, \quad \theta = 0 \ \text{ in } \widehat{\mathcal{O}} \setminus D_{R_\varepsilon}\right\}.$$

Since $f^{\infty,p}$ is $p$-positively homogeneous, by (2.5) and (6.10) there holds, for all $j \in J_\varepsilon$ and $\alpha \in \mathbb{R}$,

$$\begin{aligned}\int_{D^j_{R_\varepsilon}} f^{\infty,p}(\alpha\widehat{\nabla}\theta_\varepsilon,0)\; d\widehat{x} &= \operatorname{cap}^{f^{\infty,p}}\left(S^j_{r_\varepsilon}, D^j_{R_\varepsilon};\alpha\right) = \operatorname{cap}^{f^{\infty,p}}\left(r_\varepsilon S, R_\varepsilon D;\alpha\right)\\ &= r_\varepsilon^{2-p}\operatorname{cap}^{f^{\infty,p}}\left(S, R_\varepsilon/r_\varepsilon D;\operatorname{sgn}(\alpha)\right)|\alpha|^p.\end{aligned} \tag{5.29}$$

We set

$$u_\varepsilon(x) = \theta_\varepsilon(\widehat{x})\chi_\varepsilon(x) + (1-\theta_\varepsilon(\widehat{x}))u(x), \tag{5.30}$$

where

$$\chi_\varepsilon(x) = \sum_{j\in J_\varepsilon}\left(\fint_{S^j_{r_\varepsilon}} v(\widehat{x},x_3)\; d\widehat{x} + \left(\fint_{S^j_{r_\varepsilon}} \varphi(\widehat{x},x_3)\; d\widehat{x}\right).(\hat{x}-\omega^j_\varepsilon)\right)\mathbb{1}_{D^j_{R_\varepsilon}}(\widehat{x}). \tag{5.31}$$

It is easy to check that the convergences stated in (5.26) hold true. We have

$$\begin{aligned}&\int_{\mathcal{O}\setminus T_{r_\varepsilon}} f(\nabla u_\varepsilon)dx + \lambda_\varepsilon\int_{T_{r_\varepsilon}} g(\nabla u_\varepsilon)dx := I_{\varepsilon 1} + I_{\varepsilon 2} + I_{\varepsilon 3};\\ &I_{\varepsilon 1} = \int_{\mathcal{O}\setminus(D_{R_\varepsilon}\times(0,L))} f(\nabla u)dx,\\ &I_{\varepsilon 2} = \int_{(D_{R_\varepsilon}\times(0,L))\setminus S_{r_\varepsilon}\times(0,L)} f\left(((\chi_\varepsilon-u)\widehat{\nabla}\theta_\varepsilon,0) + (1-\theta_\varepsilon)\nabla u + \theta_\varepsilon\nabla\chi_\varepsilon\right)dx,\\ &I_{\varepsilon 3} = \lambda_\varepsilon\int_{T_{r_\varepsilon}} g\big(\nabla\chi_\varepsilon\big)\; dx.\end{aligned} \tag{5.32}$$

The proof of (5.26) is achieved provided we show that

$$\limsup_{\varepsilon\to 0} I_{\varepsilon 1} \le \int_{\mathcal{O}} f(\nabla u)dx, \tag{5.33}$$

$$\limsup_{\varepsilon\to 0} I_{\varepsilon 2} \le \int_{\mathcal{O}} c^f(S;v-u)ndx, \tag{5.34}$$

$$\lim_{\varepsilon\to 0} I_{\varepsilon 3} \le \overline{k}\int_{\mathcal{O}} g^{hom}(\partial_3 v)ndx. \tag{5.35}$$

The proof of (5.33) is straightforward.

**Proofs of (5.34).** Assuming first that $\gamma^{(p)} < +\infty$ (hence $p \le 2$ and $(\theta_\varepsilon)$ is bounded in $W^{1,p}(\widehat{\mathcal{O}})$) and applying (7.16) to $(h,A) = (f, D_{R_\varepsilon}\times(0,L)\setminus S_{r_\varepsilon}\times(0,L))$, noticing that by (1.5) there holds $|D_{R_\varepsilon}| \le C\frac{r_\varepsilon^2}{\varepsilon^2} = o(1)$ and $|\theta_\varepsilon| \le 1$ (see (6.4)), we get

$$\left| I_{\varepsilon 2} - \int_{(D_{R_\varepsilon}\times(0,L))\setminus S_{r_\varepsilon}\times(0,L)} f\left(((\chi_\varepsilon-u)\widehat{\nabla}\theta_\varepsilon,0)\right)dx\right| = o(1), \tag{5.36}$$

and then deduce from (7.8) that

$$\left| I_{\varepsilon 2} - \int_{(D_{R_\varepsilon}\times(0,L))\setminus S_{r_\varepsilon}\times(0,L)} f^{\infty,p}\big(((\chi_\varepsilon - u)\widehat{\nabla}\theta_\varepsilon, 0)\big)dx \right| = o(1). \tag{5.37}$$

It follows from (7.16), (5.37) and the estimate (see (5.31))

$$|(\chi_\varepsilon - u)(x) - (v-u)(\omega_\varepsilon^j, x_3)| \leq CR_\varepsilon \quad \text{ in } D^j_{R_\varepsilon} \times (0,L), \quad \forall j \in J_\varepsilon,$$

that

$$\left| I_{\varepsilon 2} - \sum_{j\in J_\varepsilon} \int_{(D^j_{R_\varepsilon}\times(0,L))\setminus S_{r_\varepsilon}\times(0,L)} f^{\infty,p}((v-u)(\omega_\varepsilon^j, x_3)\widehat{\nabla}\theta_\varepsilon, 0)\ dx \right| = o(1). \tag{5.38}$$

By (2.3), (5.29) and (6.10), there holds

$$\begin{aligned}\sum_{j\in J_\varepsilon} \int_{(D^j_{R_\varepsilon}\times(0,L))\setminus S^j_{r_\varepsilon}\times(0,L)} f^{\infty,p}((v-u)(\omega_\varepsilon^j, x_3)\widehat{\nabla}\theta_\varepsilon, 0)dx &= \sum_{j\in J_\varepsilon}\int_0^L \mathrm{cap}^{f^{\infty,p}}\big(S_{r_\varepsilon}, R_\varepsilon D; (v-u)(\omega^j_\varepsilon, x_3)\big)dx_3 \\ &= \frac{r_\varepsilon^{2-p}}{\varepsilon^2}\int \mathrm{cap}^{f^{\infty,p}}\big(S, R_\varepsilon/r_\varepsilon D; \zeta_\varepsilon(x)\big)d\mu_\varepsilon,\end{aligned} \tag{5.39}$$

where

$$\zeta_\varepsilon(x) := \sum_{j\in J_\varepsilon}(v-u)(\omega_\varepsilon^j, x_3)\mathbb{1}_{D^j_{R_\varepsilon}}(\widehat{x}). \tag{5.40}$$

We distinguish then two cases.
**Case $p<2$.** Let us fix some bounded open subset $V$ of $\mathbb{R}^2$ such that $\overline{S}\subset V$. For small $\varepsilon$'s there holds $V\subset R_\varepsilon/r_\varepsilon D$, hence by (6.5) $\mathrm{cap}^{f^{\infty,p}}\big(S, R_\varepsilon/r_\varepsilon D;\zeta_\varepsilon(x)\big) \leq \mathrm{cap}^{f^{\infty,p}}\big(S, V;\zeta_\varepsilon(x)\big)$, therefore by (2.1), (5.38) and (5.39) we have, since $0<\gamma^{(p)}<+\infty$,

$$\limsup_{\varepsilon\to 0} I_{\varepsilon 2} \leq \gamma^{(p)} \limsup_{\varepsilon\to 0}\int \mathrm{cap}^{f^{\infty,p}}\big(S, V;\zeta_\varepsilon(x)\big)d\mu_\varepsilon. \tag{5.41}$$

By Proposition 6.1 (i), the application $\mathrm{cap}^{f^{\infty,p}}\big(S,V;.\big)$ is locally Lipschitz continuous and by (5.40) the estimate $|\zeta_\varepsilon - (v-u)| \leq C\varepsilon$ holds true in $D_{R_\varepsilon}\times(0,L)$, because $v-u$ is continous. We deduce that $\mathrm{cap}^{f^{\infty,p}}\big(S,V;\zeta_\varepsilon(x)\big) - \mathrm{cap}^{f^{\infty,p}}\big(S,V;v-u\big) \leq C\varepsilon$ in $D_{R_\varepsilon}$ and then infer from (4.11) and (5.41) that

$$\limsup_{\varepsilon\to 0} I_{\varepsilon 2} \leq \gamma^{(p)}\int_{\mathcal{O}} \mathrm{cap}^{f^{\infty,p}}\big(S,V;v-u\big)ndx. \tag{5.42}$$

Substituting $V_n$ for $V$ in (5.42), where $(V_n)$ denotes an increasing sequence of bounded open subsets of $\mathbb{R}^2$ such that $\overline{S}\subset V_1$ and $\bigcup_{n\in\mathbb{N}} V_n = \mathbb{R}^2$, noticing that by (6.5) and (6.6) there holds $\mathrm{cap}^{f^{\infty,p}}\big(S,V_n;v-u\big) \leq \mathrm{cap}^{f^{\infty,p}}\big(S,V_1;v-u\big)$ and $\lim_{n\to+\infty}\mathrm{cap}^{f^{\infty,p}}\big(S,V_n;v-u\big) = \mathrm{cap}^{f^{\infty,p}}\big(S,\mathbb{R}^2;v-u\big)$, by applying the Dominated Convergence Theorem, we get

$$\begin{aligned}\limsup_{\varepsilon\to 0} I_{\varepsilon 2} &\leq \gamma^{(p)}\lim_{n\to+\infty}\int_{\mathcal{O}} \mathrm{cap}^{f^{\infty,p}}\big(S,V_n;v-u\big)ndx = \gamma^{(p)}\int_{\mathcal{O}} \mathrm{cap}^{f^{\infty,p}}\big(S,\mathbb{R}^2;v-u\big)ndx \\ &= \int_{\mathcal{O}} c^f(S;v-u)\ ndx. \qquad\qquad (p<2)\end{aligned} \tag{5.43}$$

The proof of (5.34) is achieved in the case $0<\gamma^{(p)}<+\infty$.

**Case** $p = 2$**.** By (5.39) and by the second line of (6.10), we have

$$\sum_{j\in J_\varepsilon}\int_{(D^j_{R_\varepsilon}\times(0,L))\setminus S^j_{r_\varepsilon}\times(0,L)} f^{\infty,p}((v-u)(\omega^j_\varepsilon,x_3)\widehat{\nabla}\theta_\varepsilon,0)\ dx$$
$$= \frac{\mathrm{cap}^{f^{\infty,2}}\big(r_\varepsilon S,R_\varepsilon D;1\big)}{\varepsilon^2}\sum_{j\in J_\varepsilon}\int|\zeta_\varepsilon|^2\mathbb{1}_{\zeta_\varepsilon>0}d\mu_\varepsilon \qquad (5.44)$$
$$+\frac{\mathrm{cap}^{f^{\infty,2}}\big(r_\varepsilon S,R_\varepsilon D;-1\big)}{\varepsilon^2}\sum_{j\in J_\varepsilon}\int|\zeta_\varepsilon|^2\mathbb{1}_{\zeta_\varepsilon<0}d\mu_\varepsilon.$$

By (2.7), (2.11), and (5.28), there holds

$$\lim_{\varepsilon\to0}\frac{\mathrm{cap}^{f^{\infty,2}}\big(r_\varepsilon S,R_\varepsilon D;\pm1\big)}{\varepsilon^2}=\gamma^{(2)}c^{f^{\infty,2}}(\pm1). \qquad (5.45)$$

By (5.40), the next estimates are satisfied

$$\begin{aligned}&\big||\zeta_\varepsilon|^2\mathbb{1}_{\zeta_\varepsilon>0}-|v-u|^2\mathbb{1}_{v-u>0}\big|\le C\varepsilon \quad \text{in } D_{R_\varepsilon}\times(0,L),\\ &\big||\zeta_\varepsilon|^2\mathbb{1}_{\zeta_\varepsilon<0}-|v-u|^2\mathbb{1}_{v-u<0}\big|\le C\varepsilon \quad \text{in } D_{R_\varepsilon}\times(0,L).\end{aligned} \qquad (5.46)$$

We deduce from (2.9), (2.11), (4.11), (5.38), (5.45), (5.46), and (6.13) that

$$\lim_{\varepsilon\to0}I_{\varepsilon2}=\gamma^{(2)}c^{f^{\infty,2}}(1)\int_{\mathcal{O}}|v-u|^2\mathbb{1}_{v-u>0}ndx+\gamma^{(2)}c^{f^{\infty,2}}(-1)\int_{\mathcal{O}}|v-u|^2\mathbb{1}_{v-u<0}ndx$$
$$=\int_{\mathcal{O}}c^f(S,v-u)\ ndx. \qquad (p=2)$$

The proof of (5.34) is achieved in the case $p=2$, $0<\gamma^{(2)}<+\infty$.
If $\gamma^{(p)}=+\infty$, we choose a sequence $(R_\varepsilon)$ satisfying, besides (2.6), the estimate

$$R^p_\varepsilon\gamma^{(p)}_\varepsilon(r_\varepsilon)<<1. \qquad (5.47)$$

By (5.25) there holds $u=v$ and by (5.31) we have $|\chi_\varepsilon-u|<CR_\varepsilon$ in $D_{R_\varepsilon}$. Taking (1.9) into account, we infer that f

$$\left|\int_{(D_{R_\varepsilon}\times(0,L))\setminus S_{r_\varepsilon}\times(0,L)} f\left(((\chi_\varepsilon-u)\widehat{\nabla}\theta_\varepsilon,0)\right)dx\right| =\le CR^p_\varepsilon\int_{\mathcal{O}}|\widehat{\nabla}\theta_\varepsilon|^pdx\le CR^p_\varepsilon\gamma^{(p)}_\varepsilon(r_\varepsilon). \qquad (5.48)$$

It follows then from (5.36), (5.48), and (5.47) that $\lim_{\varepsilon\to0}I_{\varepsilon2}=0$. □

**Proof of (5.35)**. If $\bar{k}<+\infty$, noticing that by (5.31) there holds $|\nabla\chi_\varepsilon-(\varphi_1,\varphi_2,\partial_3v)|\le cr_\varepsilon$ in $T_{r_\varepsilon}$, we deduce from (1.10), (4.11), and (5.27) that

$$\limsup_{\varepsilon\to0}\lambda_\varepsilon\int_{T_{r_\varepsilon}}g(\nabla\chi_\varepsilon)\ dx=\limsup_{\varepsilon\to0}\lambda_\varepsilon\frac{r^2_\varepsilon|S|}{\varepsilon^2}\int g(\varphi,\partial_3v)\ d\mu_\varepsilon=\bar{k}\int_{\mathcal{O}}g(\varphi,\partial_3v)ndx$$
$$=\bar{k}\int_{\mathcal{O}}g^{hom}(\partial_3v)\ ndx.$$

Otherwise, if $\bar{k}=+\infty$, then by (5.25) we have $\partial_3v=0$, therefore $\varphi=0$ (because by (1.9) there holds $g(0)=0$) and $\chi_\varepsilon=0$. Accordingly, $I_{\varepsilon3}=0$ and (5.35) is proved. □

□

# 6 Some properties of $f$-capacities

Our main objective in this section is to analyze the behavior of the application $\text{cap}^f$ defined by (2.5) with respect to certain small subsets of $\mathbb{R}^2$. This analyzis reveals striking differences depending on the rate of growth $p$ of the function $f$. These disparities originate mainly in the fact that Gagliardo-Nirenberg-Sobolev inequality in $\mathbb{R}^2$ [8, Theorem 9.9], namely

$$\int_{\mathbb{R}^2} |f|^{p^*} dx \leq C \int_{\mathbb{R}^2} |\nabla f|^p dx \quad \forall f \in W^{1,p}(\mathbb{R}^2) \quad p^* := \frac{Np}{N-p}, \tag{6.1}$$

fails to hold for $p \geq 2$. If $1 \leq p < 2$, then by (7.10), for any open subset $V$ of $\mathbb{R}^2$, the application $\varphi \to \left(\int_V |\nabla\varphi|^p dx\right)^{\frac{1}{p}}$ is a norm on $W_0^{1,p}(V)$. If $1 < p < N$, the completion of $W_0^{1,p}(V)$ with respect to this norm is the reflexive Banach space defined by

$$K_0^p(V) := \left\{ f \in L^{p^*}(\mathbb{R}^2), \quad \nabla f \in L^p(\mathbb{R}^2) \right\}. \tag{6.2}$$

The space $K_0^p(V)$ is equal to $W_0^{1,p}(V)$ if $V$ is bounded and may be strictly larger otherwise. If $1 < p < 2$ and if $U$ is bounded, the infimum problem (2.5) is achieved in the space $K_0^p(V)$ for whatever choice of $V$, whereas if $2 \leq p$, it is not achieved in any Banach space of functions if $V = \mathbb{R}^2$ (nor, in general, if $V$ is unbounded). This lack of solution, similar to the Stoke's paradox in fluid mechanics [19], marks a fundamental difference between the cases $1 < p < 2$ and $2 \leq p$.

A series of properties of the application $\text{cap}^f$ is collected in the next proposition. Further results concerning $f$-capacities and many references on this subject may be found for instance in [16].

**Proposition 6.1.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly convex mapping satisfying the growth condition (1.9) for some $p \in (1, +\infty)$, let $S$ be a bounded connected Lipschitz open subset of $\mathbb{R}^2$, and let $V$ be an open subset of $\mathbb{R}^2$ such that $0 \in S \subset \overline{S} \subset V$. Then,*
**(i)** *The application $\alpha \in \mathbb{R} \to \text{cap}^f(S, V; \alpha)$ is convex.*
**(ii)** *If $1 < p < 2$, then*

$$\text{cap}^f(S, V; \alpha) := \min\left\{ \int_V f(\widehat{\nabla}\psi) dx, \ \psi \in K_0^p(V), \ \psi = \alpha \text{ in } S \right\}. \tag{6.3}$$

*Moreover, the solution $\psi$ to (6.3) is unique and satisfies, for a.e. $x \in \mathcal{O}$,*

$$0 \leq \psi(x) \leq \alpha \quad \text{if} \quad \alpha \geq 0; \qquad 0 \geq \psi(x) \geq \alpha \quad \text{if} \quad \alpha \leq 0. \tag{6.4}$$

**(iii)** *Let $V_1$, $V_2$ be two open subsets of $\mathbb{R}^2$. Then*

$$\overline{S} \subset V_1 \subset V_2 \quad \Rightarrow \quad \text{cap}^f(S, V_1; \alpha) \geq \text{cap}^f(S, V_2; \alpha) \quad \forall \alpha \in \mathbb{R}. \tag{6.5}$$

*Moreover, if $(V_n)$ is an increasing sequence of open subsets of $\mathbb{R}^2$ such that $\overline{S} \subset V_1$ and $\bigcup_{n=1}^{+\infty} V_n = V$, then*

$$\lim_{n \to +\infty} \text{cap}^f(S, V_n; \alpha) = \text{cap}^f(S, V; \alpha) \quad \forall \alpha \in \mathbb{R}. \tag{6.6}$$

*Furthermore, there holds*

$$\lim_{\lambda \to 0} \text{cap}^f\left(S; \frac{1}{\lambda} V; \alpha\right) = \text{cap}^f\left(S; \mathbb{R}^2; \alpha\right). \tag{6.7}$$

*Assume in addition that $1 < p < 2$ or that $V$ is bounded, and let $\psi_n$ be the solution to the problem deduced from (6.3) by substituting $V_n$ for $V$. Then the sequence $(\psi_n)$, where $\psi_n$ is extended to $V$ by setting $\psi_n = 0$ in $V \setminus V_n$, converges weakly in $K_0^p(V)$ to the unique solution to (6.3).*
**(iv)** *Let $S_1$ and $S_2$ be two bounded open subsets of $\mathbb{R}^2$ such that $\overline{S}_1 \subset S_2 \subset \overline{S}_2 \subset V$. Then*

$$\text{cap}^f(S_1, V; \alpha) \leq \text{cap}^f(S_2, V; \alpha) \quad \forall \alpha \in \mathbb{R}. \tag{6.8}$$

*If $p < 2$ and if $(S_n)$ is an increasing sequence of bounded open subsets of $\mathbb{R}^2$ such that $\bigcup_{n=1}^{+\infty} S_n = S$, then*

$$\lim_{n\to+\infty} \operatorname{cap}^f(S_n, V; \alpha) = \operatorname{cap}^f(S, V; \alpha) \quad \forall \alpha \in \mathbb{R}. \tag{6.9}$$

**(v)** *Assume that $f$ is $p$-positively homogeneous and let $\lambda > 0$, $\alpha \in \mathbb{R}$. Then*

$$\begin{aligned} &\operatorname{cap}^f(\lambda S, V; \alpha) = \lambda^{2-p} \operatorname{cap}^f\left(S, \frac{1}{\lambda} V; \alpha\right) \qquad \text{if} \quad \lambda S \subset V, \\ &\operatorname{cap}^f(S, V; \alpha) = |\alpha|^p \operatorname{cap}^f(S, V; \operatorname{sgn}(\alpha)). \end{aligned} \tag{6.10}$$

**(vi)** *There holds*

$$\operatorname{cap}^{\frac{1}{p}|\cdot|^p}\big(r_\varepsilon D, R_\varepsilon D; a\big) = \begin{cases} \dfrac{2\pi}{p}\Big(\dfrac{s}{R_\varepsilon^s - r_\varepsilon^s}\Big)^{p-1} |a|^p, \;\; s = \dfrac{p-2}{p-1} & \text{if } p \neq 2, \\ \frac{\pi}{\ln(R_\varepsilon/r_\varepsilon)} a^2 & \text{if } p = 2. \end{cases} \tag{6.11}$$

**(vii)** *We have*

$$\begin{aligned} &\operatorname{cap}^f(S, \mathbb{R}^2; \alpha) > 0 \quad \forall \alpha \in \mathbb{R} \setminus \{0\} && \text{if} \quad 1 < p < 2, \\ &\operatorname{cap}^f(S, \mathbb{R}^2; \alpha) = 0 \quad \forall \alpha \in \mathbb{R} && \text{if} \quad 2 \leq p < +\infty. \end{aligned} \tag{6.12}$$

**(viii)** *Let $(r_\varepsilon)$ and $(R_\varepsilon)$ be two sequences of positive reals such that $r_\varepsilon << R_\varepsilon << \varepsilon$. Then*

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \operatorname{cap}^f(r_\varepsilon S, R_\varepsilon D; \alpha) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \operatorname{cap}^{f^{\infty,2}}(r_\varepsilon S, R_\varepsilon D; \alpha), \tag{6.13}$$

*provided one of these limits exists. Moreover,*

$$\begin{aligned} &\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \operatorname{cap}^{f^{\infty,p}}(r_\varepsilon S, R_\varepsilon D; \alpha) = \gamma^{(p)} \operatorname{cap}^{f^{\infty,p}}(S, \mathbb{R}^2; \alpha) \quad \text{if} \;\; 1 < p < 2, \\ &\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \operatorname{cap}^{f^{\infty,p}}(r_\varepsilon S, R_\varepsilon D; \alpha) = +\infty \qquad\qquad \text{if} \;\; 2 < p < +\infty, \end{aligned} \tag{6.14}$$

*where $\gamma^{(p)}$ is defined by (2.1).*
**(ix)** *Assume that $p = 2$ and let $c^{f^{\infty,2}}(\pm 1)$ be defined by (2.8), (2.11). Then for all sequence $(R_\varepsilon)$ satisfying (2.6) and for all bounded domain $S'$ of $\mathbb{R}^2$, there holds*

$$\lim_{k\to+\infty} \frac{\operatorname{cap}^{f^{\infty,p}}(r_{\varepsilon_k} S', R_{\varepsilon_k} D; \pm 1)}{{\varepsilon_k}^2} = \gamma^{(2)} c^{f^{\infty,2}}(\pm 1) \tag{6.15}$$

*Proof.* **(i)** Let $(\alpha, \alpha') \in \mathbb{R}^2$, $\lambda \in (0,1)$, $t > 0$ and $\eta \in \mathcal{D}(V)$ (resp. $\eta' \in \mathcal{D}(V)$) satisfying the boundary condition associated with the problem $\operatorname{cap}^f(S, V; \alpha)$ (resp. $\operatorname{cap}^f(S, V; \alpha')$) and such that

$$\int_V f(\widehat{\nabla}\eta) dx \leq \operatorname{cap}^f(S, V; \alpha) + t \quad \left(\text{resp.} \quad \int_V f(\widehat{\nabla}\eta') dx \leq \operatorname{cap}^f(S, V; \alpha') + t\right).$$

Then $\lambda\eta + (1-\lambda)\eta'$ satisfies the boundary condition associated with the problem $\operatorname{cap}^f(S, V; \lambda\alpha + (1-\lambda)\alpha')$ and

$$\begin{aligned} \operatorname{cap}^f(S, V; \lambda\alpha + (1-\lambda)\alpha') &\leq \int_V f(\widehat{\nabla}(\lambda\eta + (1-\lambda)\eta')) dx \\ &\leq \lambda \int_V f(\widehat{\nabla}\eta) dx + (1-\lambda) \int_V f(\widehat{\nabla}\eta') dx \\ &\leq \lambda \operatorname{cap}^f(S, V; \alpha) + (1-\lambda) \operatorname{cap}^f(S, V; \alpha') + t. \end{aligned}$$

**(ii)** If $1 < p < 2$ or if $V$ is bounded, by the density of $\mathcal{D}(V)$ in $K_0^p(V)$ and the strong continuity on $K_0^p(V)$ of the application $\psi \to \int_V f(\widehat{\nabla}\psi)dx$, we have

$$\operatorname{cap}^f(S, V; \alpha) = \inf\left\{ \int_V f(\widehat{\nabla}\psi)dx,\ \psi \in K_0^p(V),\ \psi = \alpha \text{ in } S \right\}. \tag{6.16}$$

Let $(\psi_\varepsilon)$ be a sequence of minimizers of (6.16). By (1.9) and Korn inequality, we can assume that $\sup_{\varepsilon>0} \int_V |\nabla\psi_\varepsilon|^p dx < +\infty$ hence, if $1 < p < 2$ or if $V$ is bounded, then $(\psi_\varepsilon)$ is bounded in the reflexive Banach space $K_0^p(V)$ and converges weakly, up to a subsequence, to some $\psi \in K_0^p(V)$. It is easy to check that $\psi = \alpha$ in $S$. The application $\psi \to \int_V f(\widehat{\nabla}\psi)dx$ being convex and strongly continuous on $K_0^p(V)$, it is also weakly lower semi-continuous, therefore

$$\operatorname{cap}^f(S, V; \alpha) = \lim_{\varepsilon\to 0} \int_V f(\widehat{\nabla}\psi_\varepsilon)dx \geq \int_V f(\widehat{\nabla}\psi)dx \geq \operatorname{cap}^f(S, V; \alpha).$$

Therefore the infimum problem (6.16) is achieved. The uniqueness of its solution results from the strict convexity of $f$. Assertion (6.4) follows from the last mentioned uniqueness and from the following property if $\alpha > 0$

$$\int_V f((\psi \vee 0) \wedge \alpha)dx \leq \int_V f(\psi)dx \qquad \forall \psi \in K_0(V),$$

and from a similar one if $\alpha < 0$.

**(iii)** The assertion (6.5) is straightforward. To prove (6.6), we fix $t > 0$, $\psi \in \mathcal{D}(V)$ such that $\psi = \alpha$ in $S$ and $\int_V f(\widehat{\nabla}\psi)dx \leq \operatorname{cap}^f(S, V; \alpha) + t$ and $n_0 \in \mathbb{N}$ such that $\operatorname{spt}\psi \subset V_{n_0}$. We have $\operatorname{cap}^f(S, V_n; \alpha) \leq \int_V f(\widehat{\nabla}\psi)dx$ $\forall n \geq n_0$, hence

$$\begin{aligned}\operatorname{cap}^f(S, V; \alpha) &\leq \liminf_{n\to+\infty} \operatorname{cap}^f(S, V_n; \alpha) \\ &\leq \limsup_{n\to+\infty} \operatorname{cap}^f(S, V_n; \alpha) \leq \operatorname{cap}^f(S, V; \alpha) + t.\end{aligned}$$

Assertion (6.6) is proved.
Since $0 \in V$, we can assume without loss of generality that $D \subset V$. By (6.6) we have $\lim_{\lambda\to 0} \operatorname{cap}^f\left(S, \frac{1}{\lambda}D; \alpha\right) = \operatorname{cap}^f(S, \mathbb{R}^2; \alpha)$. By passing to the limit as $\lambda \to 0$ in the first and third terms of the double inequality $\operatorname{cap}^f(S, \mathbb{R}^2; \alpha) \leq \operatorname{cap}^f(S, \frac{1}{\lambda}V; \alpha) \leq \operatorname{cap}^f(S, \frac{1}{\lambda}D; \alpha)$ we obtain (6.7).
If $1 < p < 2$, then by (1.9) and (6.6) we have

$$|\psi_n|^p_{K_0^p(V)} \leq C\operatorname{cap}^f(S, V_n; \alpha) \leq C(\operatorname{cap}^f(S, V; \alpha) + 1) < +\infty,$$

hence the sequence $(\psi_n)$ is bounded in $K_0^p(V)$ and converges weakly, up to a subsequence, to some $\psi \in K_0^p(V)$. Taking (6.6) into account, we deduce that

$$\begin{aligned}\operatorname{cap}^f(S, V; \alpha) \geq \lim_{n\to+\infty} \operatorname{cap}^f(S, V_n; \alpha) &= \lim_{n\to+\infty} \int_V f(\widehat{\nabla}\psi_n)dx \\ &\geq \int_V f(\widehat{\nabla}\psi)dx \geq \operatorname{cap}^f(S, V; \alpha).\end{aligned}$$

**(iv)** The assertion (6.8) is straightforward. Let us denote by $\psi_n$ the unique solution to the problem deduced from (6.3) by substituting $S_n$ for S. Then $(\psi_n)$ is bounded in the reflexive Banach space $K_0^p(V)$ and converges weakly, up to a subsequence, to some $\psi \in K_0^p(V)$. It is easy to check that $\psi = \alpha$ a.e. in $S$, therefore by the lower weak semicontinuity of the application $\varphi \to \int_V f(\nabla\varphi)dx$, we get

$$\liminf_{n\to+\infty} \operatorname{cap}^f(S_n, V; \alpha) = \liminf_{\varepsilon\to 0} \int_V f(\nabla\psi_n)dx \geq \int_V f(\nabla\psi)dx \geq \operatorname{cap}^f(S, V; \alpha). \tag{6.17}$$

Conversely, by (6.8) we have $\limsup_{n\to+\infty} \operatorname{cap}^f(S_n, V; \alpha) \leq \operatorname{cap}^f(S, V; \alpha)$. Assertion (6.9) is proved.

**(v)** Let us fix $t > 0$ and let $\psi \in \mathcal{D}(V)$ satisfying the boundary conditions associated with $\mathrm{cap}^f(\lambda S, V; \alpha)$ such that $\mathrm{cap}^f(\lambda S, V; \alpha) + t \geq \int_V f(\widehat{\nabla}\psi))dx$. Then the field $\varphi \in \mathcal{D}\left(\frac{1}{\lambda}V\right)$ defined by setting $\varphi(y) := \psi(\lambda y)$ satisfies $\varphi = \alpha$ in $S$ and $\widehat{\nabla}\varphi(y) = \lambda\widehat{\nabla}\psi(\lambda y)$, therefore by the change of variables formula, since $f$ is assumed to be $p$-positively homogeneous, we have

$$\begin{aligned}\mathrm{cap}^f(\lambda S, V; \alpha) + t &\geq \int_V f(\widehat{\nabla}\psi)dx = \lambda^2 \int_{\frac{1}{\lambda}V} f(\widehat{\nabla}\psi)(\lambda y)dy \\ &= \lambda^{2-p} \int_{\frac{1}{\lambda}V} f(\widehat{\nabla}\varphi)(y)dy \geq \lambda^{2-p}\mathrm{cap}^f\left(S, \frac{1}{\lambda}V; \alpha\right).\end{aligned}$$

By the arbitrary choice of $\lambda$, $S$, $V$, the first line of (6.10) is proved.

**(vi)** The solution to the problem associated with $\mathrm{cap}^{\frac{1}{p}|\cdot|^p}\big(r_\varepsilon D, R_\varepsilon D; a\big)$ is radial, hence can be easily computed by solving an elementary one dimensional problem, yielding (6.11) (see [5, p. 432]).

**(vii)** If $1 < p < 2$, then by (6.3) there exists a solution $\psi \in K_0^p(V)$ to the problem associated with $\mathrm{cap}^f(S, \mathbb{R}^2; \alpha)$. This function $\psi$ is not equal to zero, because $\psi = \alpha$ in $S$, hence by (1.9) there holds $\mathrm{cap}^f(S, \mathbb{R}^2; \alpha) = \int_V f(\nabla\psi)dx \geq C\int_V |\nabla\psi|^p dx > 0$, because the application $\varphi \to \left(\int_V |\nabla\varphi|^p dx\right)$ is a norm on $K_0^p(V)$.

If $p \geq 2$, fixing $r > 0$ such that $S \subset rD$, we deduce from (1.9), (6.7), (6.8), and (6.11) that

$$\mathrm{cap}^f(S, \mathbb{R}^3; \alpha) \leq C\mathrm{cap}^{\frac{1}{p}|\cdot|^p}(S, \mathbb{R}^3; \alpha) \leq C\mathrm{cap}^{\frac{1}{p}|\cdot|^p}(rD, \mathbb{R}^3; \alpha) = \lim_{R\to+\infty} C\mathrm{cap}^{\frac{1}{p}|\cdot|^p}(rD, RD; \alpha) = 0.$$

**(viii)** By (1.9) and (6.11) there holds, for $h \in \{f, f^{\infty,p}\}$,

$$C\gamma_\varepsilon^{(p)}(r_\varepsilon) \geq \frac{C}{\varepsilon^2}\mathrm{cap}^{|\cdot|^p}(r_\varepsilon S, R_\varepsilon D; \alpha) \geq \frac{1}{\varepsilon^2}\mathrm{cap}^h(r_\varepsilon S, R_\varepsilon D; \alpha) \geq \frac{C}{\varepsilon^2}\mathrm{cap}^{|\cdot|^p}(r_\varepsilon S, R_\varepsilon D; \alpha) \geq C\gamma_\varepsilon^{(p)}(r_\varepsilon),$$

therefore if $\gamma^{(p)} \in \{0, +\infty\}$ (and in particular if $p > 2$) there is nothing to prove.

Assume that $0 < \gamma^{(p)} < +\infty$ and let $\varphi$ denote the solution to the problem associated with $\mathrm{cap}^f(r_\varepsilon S, R_\varepsilon D; \alpha)$ (see (ii)). By (1.9) and (6.11), there holds

$$\int_{R_\varepsilon D} |\nabla\varphi|^p dx \leq C\mathrm{cap}^f(r_\varepsilon S, R_\varepsilon D; \alpha) \leq C\mathrm{cap}^{|\cdot|^p}(r_\varepsilon S, R_\varepsilon D; \alpha) \leq \begin{cases} Cr_\varepsilon^{2-p} & \text{if } p \neq 2, \\ \frac{C}{|\log r_\varepsilon|} & \text{if } p = 2. \end{cases} \tag{6.18}$$

By (2.10) we have

$$\begin{aligned}\Delta_\varepsilon &:= \frac{1}{\varepsilon^2}\mathrm{cap}^{f^{\infty,p}}(r_\varepsilon S, R_\varepsilon D; \alpha) - \frac{1}{\varepsilon^2}\mathrm{cap}^f(r_\varepsilon S, R_\varepsilon D; \alpha) \leq \frac{1}{\varepsilon^2}\int_{R_\varepsilon D} f^{\infty,p}(\nabla\varphi) - f(\nabla\varphi)dx \\ &\leq \alpha'\frac{1}{\varepsilon^2}\int_{R_\varepsilon D}(1 + |\nabla\varphi|^{\beta'})dx \leq C\frac{R_\varepsilon^2}{\varepsilon^2} + C\frac{1}{\varepsilon^2}\left(\int_{R_\varepsilon D}|\nabla\varphi|^p dx\right)^{\frac{\beta'}{p}}\left(R_\varepsilon^2\right)^{1-\frac{\beta'}{p}}.\end{aligned} \tag{6.19}$$

We deduce from (6.18) and (6.19) that

$$\begin{aligned}\Delta_\varepsilon &\leq C\frac{R_\varepsilon^2}{\varepsilon^2} + C\frac{1}{\varepsilon^2}\left(r_\varepsilon^{2-p}\right)^{\frac{\beta'}{p}}\left(R_\varepsilon^2\right)^{1-\frac{\beta'}{p}} \leq C\frac{R_\varepsilon^2}{\varepsilon^2} + C\gamma_\varepsilon(r_\varepsilon)\left(R_\varepsilon^2 r_\varepsilon^{2-p}\right)^{1-\frac{\beta'}{p}} = o(1) \qquad \text{if } p < 2, \\ \Delta_\varepsilon &\leq C\frac{R_\varepsilon^2}{\varepsilon^2} + C\frac{1}{\varepsilon^2\left(|\log(r_\varepsilon)|\right)^{\frac{\beta'}{p}}}\left(R_\varepsilon^2\right)^{1-\frac{\beta'}{p}} \leq C\frac{R_\varepsilon^2}{\varepsilon^2} + C\gamma_\varepsilon(r_\varepsilon)\left(R_\varepsilon^2|\log r_\varepsilon|\right)^{1-\frac{\beta'}{p}} = o(1) \quad \text{if } p = 2,\end{aligned}$$

(because if $p = 2$ and $0 < \gamma^{(2)} < +\infty$, then $R_\varepsilon^2|\log r_\varepsilon| \ll \varepsilon^2|\log r_\varepsilon| = O(1)$). In the same manner, we find that $\frac{1}{\varepsilon^2}\mathrm{cap}^f(r_\varepsilon S, R_\varepsilon D; \alpha) - \frac{1}{\varepsilon^2}\mathrm{cap}^{f^{\infty,p}}(r_\varepsilon S, R_\varepsilon D; \alpha) = o(1)$. The assertion (6.13) and the second line of

(6.14) are proved. Since $f^{\infty,p}$ is positively homogeneous of order $p$, we infer from (2.1), (6.6) and (6.10) that

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\mathrm{cap}^{f^{\infty,p}}(r_\varepsilon S, R_\varepsilon D;\alpha) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\mathrm{cap}^{f}(r_\varepsilon S, R_\varepsilon D;\alpha)\\ &= \lim_{\varepsilon\to 0}\frac{r_\varepsilon^{2-p}}{\varepsilon^2}\mathrm{cap}^{f}(S, R_\varepsilon/r_\varepsilon D;\alpha) = \gamma^{(p)}\mathrm{cap}^{f}(S,\mathbb{R}^2;\alpha).\end{aligned}$$

The first line of (6.14) is proved.

**(ix)** First we assume that $S = S'$ and prove (6.15) by arguing by contradiction. Otherwise, there exists a sequence $(R_\varepsilon)$ satisfying (2.6) and a subsequence $(\varepsilon_{k_l})$ of $(\varepsilon_k)$ such that

$$\lim_{l\to+\infty} C_{\varepsilon_{k_l}}(R_{\varepsilon_{k_l}},\pm 1) = \gamma^{(2)}c(\pm 1), \tag{6.20}$$

for some couple $(c(1), c(-1))$ different from $(c^{f^{\infty,2}}(1), c^{f^{\infty,2}}(-1))$. By substituting the assumption (6.20) for the assumption (2.8) in propositions 5.1, 5.2, we obtain the assertions deduced from (5.5), (5.24) by substituting $\varepsilon_{k_l}$ for $\varepsilon$ and $\Phi_c$ for $\Phi$, where $\Phi_c$ is deduced from $\Phi$ by replacing $c^{f^{\infty,2}}(\pm 1)$ by $c(\pm 1)$ in (2.11), (2.12). As $F_{\varepsilon_{k_l}}$ is extracted from $F_{\varepsilon_k}$, the same bounds are also satisfied with $\Phi$ in place of $\Phi_c$. We deduce that $\Phi = \Phi_c$, hence $(c(1), c(-1)) = (c^{f^{\infty,2}}(1), c^{f^{\infty,2}}(-1))$, yielding a contradiction.
If $S' \neq S$, then by (1.7) there exists a couple of positive reals $(r_1, r_2)$ such that $r_1 S \subset S' \subset r_2 S$, so that by (6.8) and (6.10), there holds

$$\frac{\mathrm{cap}^{f^{\infty,p}}(r_{\varepsilon_k}S, R_{\varepsilon_k}/r_1 D;\pm 1)}{{\varepsilon_k}^2} \leq \frac{\mathrm{cap}^{f^{\infty,p}}(r_{\varepsilon_k}S', R_{\varepsilon_k}D;\pm 1)}{{\varepsilon_k}^2} \leq \frac{\mathrm{cap}^{f^{\infty,p}}(r_{\varepsilon_k}S, R_{\varepsilon_k}/r_2 D;\pm 1)}{{\varepsilon_k}^2}. \tag{6.21}$$

Then we pass to the limit as $k\to+\infty$ in the terms of the first and third terms of the double inequality (6.21).

□

# 7 Appendix: some technical lemmas related to the lower bound

**Lemma 7.1.** *Let $(u_\varepsilon)$ be a sequence satisfying (4.1), (4.6) and (4.7). Then there exists a sequence $(\widehat{u}_\varepsilon)$ verifying (5.9).*

*Proof.* We fix two sequences $(a_\varepsilon)$ and $(b_\varepsilon)$ of positive reals such that

$$1 \gg a_\varepsilon \gg b_\varepsilon, \qquad a_\varepsilon b_\varepsilon^2 \gg \frac{R_\varepsilon^2}{\varepsilon^2}. \tag{7.1}$$

By means of De Giorgi's slicing argument (see Remark 7.1), we can choose for each $\varepsilon$ a finite sequence $(l_{k,\varepsilon})_{k\in\{1,\dots,m_\varepsilon\}}$ such that $0 = l_{0,\varepsilon} < l_{1,\varepsilon} < \cdots < l_{m_\varepsilon,\varepsilon} < l_{m_\varepsilon+1,\varepsilon} = L$ and

$$\begin{aligned}&\left(k-\frac{1}{4}\right)a_\varepsilon \leq l_{k,\varepsilon} \leq \left(k+\frac{1}{4}\right)a_\varepsilon, \quad m_\varepsilon \sim \frac{L}{a_\varepsilon},\\ &\int_{H_\varepsilon}|\nabla u_\varepsilon|^p dx \leq C\frac{b_\varepsilon}{a_\varepsilon}\int_{\mathcal{O}}|\nabla u_\varepsilon|^p dx \quad (= o(1)),\\ &\int_{H_\varepsilon}|\langle u_\varepsilon\rangle_{r_\varepsilon}|^p + |\langle u_\varepsilon\rangle_{R_\varepsilon}|^p d\mu_\varepsilon \leq C\frac{b_\varepsilon}{a_\varepsilon}\int |\langle u_\varepsilon\rangle_{r_\varepsilon}|^p + |\langle u_\varepsilon\rangle_{R_\varepsilon}|^p d\mu_\varepsilon \quad (= o(1)),\\ &H_\varepsilon := D_{R_\varepsilon}\times\bigcup_{k=1}^{m_\varepsilon}\left(l_{k,\varepsilon}-\frac{1}{2}b_\varepsilon; l_{k,\varepsilon}+\frac{1}{2}b_\varepsilon\right)\cap\mathcal{O}.\end{aligned} \tag{7.2}$$

Then, given a sequence $(\varphi_\varepsilon) \subset \mathcal{D}(0,L)$ such that

$$\begin{aligned}
&\varphi_\varepsilon = 1 \quad \text{in} \quad (0,L) \setminus \bigcup_{k=1}^{m_\varepsilon} \left( l_{k,\varepsilon} - \frac{1}{2} b_\varepsilon ; l_{k,\varepsilon} + \frac{1}{2} b_\varepsilon \right), \quad \varphi_\varepsilon = 0 \quad \text{on} \quad \bigcup_{k=0}^{m_\varepsilon+1} \{l_{k,\varepsilon}\}, \\
&0 \leq \varphi_\varepsilon \leq 1, \quad |\varphi_\varepsilon'| < \frac{C}{b_\varepsilon},
\end{aligned} \tag{7.3}$$

we set

$$\widehat{u}_\varepsilon(\widehat{x}, x_3) := \sum_{k=1}^{m_\varepsilon+1} \left( \fint_{(l_{k-1,\varepsilon}; l_{k,\varepsilon})} \varphi_\varepsilon(s_3) u_\varepsilon(\widehat{x}, s_3) ds_3 \right) \mathbb{1}_{(l_{k-1,\varepsilon}; l_{k,\varepsilon})}(x_3), \tag{7.4}$$

and claim that the sequence $(\widehat{u}_\varepsilon)$ defined by (7.4) satisfies (5.9).

By Jensen's inequality we have, since $0 \leq \varphi_\varepsilon \leq 1$,

$$\begin{aligned}
\int_{T_{r_\varepsilon}} |\widehat{\nabla} \widehat{u}_\varepsilon|^p dx &= \sum_{k=1}^{m_\varepsilon+1} \int_{S_{r_\varepsilon}} \int_{(l_{k-1,\varepsilon}; l_{k,\varepsilon})} \left| \fint_{(l_{k-1,\varepsilon}; l_{k,\varepsilon})} \varphi_\varepsilon(s_3) \widehat{\nabla} u_\varepsilon(\widehat{x}, s_3) \right|^p ds_3 dx \\
&= \sum_{k=1}^{m_\varepsilon+1} \int_{S_{r_\varepsilon}} \int_{(l_{k-1,\varepsilon}; l_{k,\varepsilon})} \fint_{(l_{k-1,\varepsilon}; l_{k,\varepsilon})} \left| \widehat{\nabla} u_\varepsilon(\widehat{x}, s_3) \right|^p ds_3 dx \leq \int_{T_{r_\varepsilon}} |\nabla u_\varepsilon|^p dx,
\end{aligned}$$

which proves the first inequality of the first line of (5.9). The second one is obtained in the same way.

The second line of (5.9) is a consequence of (4.6) and (7.4). By applying Lemma 4.2, taking (4.11) into account, we infer that the sequence of measures $(\langle \widehat{u}_\varepsilon \rangle_{r_\varepsilon} \mu_\varepsilon)$ weak-$*$ converges to $wn\mathcal{L}^3_{\lfloor \mathcal{O}}$ for some $w \in L^p(\mathcal{O})$. Hence we just have to prove that $wn = vn \ \mathcal{L}^3_{\lfloor \mathcal{O}}$ a.e.. To that aim, we first notice that, by (7.2), the sequence $(\langle u_\varepsilon \rangle_{r_\varepsilon} \mathbb{1}_{H_\varepsilon} \mu_\varepsilon)$ weak-$*$ converges to 0 in $\mathcal{M}_b(\overline{\mathcal{O}})$. Since the support of $(1-\varphi_\varepsilon)\langle u_\varepsilon \rangle_{r_\varepsilon}$ is included in $H_\varepsilon$ and $0 \leq \varphi_\varepsilon \leq 1$ (see (7.3)), we deduce from (4.7) that $(\varphi_\varepsilon \langle u_\varepsilon \rangle_{r_\varepsilon} \mu_\varepsilon)$ weak-$*$ converges in $\mathcal{M}_b(\overline{\mathcal{O}})$ to $vn\mathcal{L}^3_{\lfloor \mathcal{O}}$. Let us fix $\psi \in C(\overline{\mathcal{O}})$ and set

$$\overline{\psi}_\varepsilon(x) := \sum_{k=1}^{m_\varepsilon+1} \left( \fint_{(l_{k-1,\varepsilon}; l_{k,\varepsilon})} \psi(\widehat{x}, s_3) ds_3 \right) \mathbb{1}_{(l_{k-1,\varepsilon}; l_{k,\varepsilon})}(x_3). \tag{7.5}$$

It is easy to check that $|\psi - \overline{\psi}_\varepsilon|_{L^\infty(\mathcal{O})} \leq C a_\varepsilon << 1$ (see (7.1), (7.2)), therefore by the two last mentioned convergences, we have

$$\lim_{\varepsilon \to 0} \int \overline{\psi}_\varepsilon \varphi_\varepsilon \langle u_\varepsilon \rangle_{r_\varepsilon} d\mu_\varepsilon = \int_{\mathcal{O}} \psi v n dx, \qquad \lim_{\varepsilon \to 0} \int \overline{\psi}_\varepsilon \langle \widehat{u}_\varepsilon \rangle_{r_\varepsilon} d\mu_\varepsilon = \int_{\mathcal{O}} \psi w n dx. \tag{7.6}$$

On the other hand there holds, by (2.3), (4.2), (7.4) and (7.5),

$$\begin{aligned}
\int \overline{\psi}_\varepsilon \varphi_\varepsilon \langle u_\varepsilon \rangle_{r_\varepsilon} d\mu_\varepsilon &= \sum_{k=1}^{m_\varepsilon+1} \frac{\varepsilon^2}{r_\varepsilon^2 |S|} \int_{S_{r_\varepsilon}} d\hat{x} \int_{(l_{k-1,\varepsilon}; l_{k,\varepsilon})} \left( \fint_{(l_{k-1,\varepsilon}; l_{k,\varepsilon})} \psi(\widehat{x}, s_3) ds_3 \right) \varphi_\varepsilon \langle u_\varepsilon \rangle_{r_\varepsilon} dx_3 \\
&= \sum_{k=1}^{m_\varepsilon+1} \frac{\varepsilon^2}{r_\varepsilon^2 |S|} \int_{S_{r_\varepsilon}} d\hat{x} \int_{(l_{k-1,\varepsilon}; l_{k,\varepsilon})} dx_3 \left( \fint_{(l_{k-1,\varepsilon}; l_{k,\varepsilon})} \psi(\widehat{x}, s_3) ds_3 \right) \left( \fint_{(l_{k-1,\varepsilon}; l_{k,\varepsilon})} \varphi_\varepsilon \langle u_\varepsilon \rangle_{r_\varepsilon} ds_3 \right) \\
&= \int \overline{\psi}_\varepsilon \langle \widehat{u}_\varepsilon \rangle_{r_\varepsilon} d\mu_\varepsilon.
\end{aligned} \tag{7.7}$$

By the arbitrary choice of $\psi$, we deduce from (7.6) and (7.7) that $nv = nw$ a.e. in $\mathcal{O}$. The first convergence of the third line of (5.9) is proved. The proof of the second one is similar.

By (2.10) and Hölder's inequality, for any measurable subset $A \subset \mathbb{R}^3$, there holds

$$\left| \int_A f^{\infty,p}(\varphi) \ dx - \int_A f(\varphi) \ dx \right| \leq \alpha' \left( |A| + |A|^{1-\frac{\beta'}{p}} |\varphi|^{\frac{\beta'}{p}}_{L^p(A)} \right) \qquad \forall \varphi \in L^p(A). \tag{7.8}$$

By (2.10) and (7.8), we have

$$\liminf_{\varepsilon\to 0}\int_{(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}} f(\nabla u_\varepsilon)\,dx=\liminf_{\varepsilon\to 0}\int_{(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}} f^{\infty,p}(\nabla u_\varepsilon)\,dx. \tag{7.9}$$

By the continuous embedding of $W^{1,p}(\mathcal{O})$ into $L^{p^*}(\mathcal{O})$ (see [8, Corollary 9.14]), we have

$$\left(\int_{\mathcal{O}}|u_\varepsilon(x)|^{p^*}\,dx\right)^{\frac{1}{p^*}}\le C|u_\varepsilon|_{W^{1,p}(\mathcal{O})}\le C,\qquad \frac{1}{p^*}=\frac{1}{p}-\frac{1}{3}. \tag{7.10}$$

Taking (1.9), (7.1), (7.2), (7.3), and (7.10) into account, applying Hölder's inequality and noticing that by (7.2) there holds $|H_\varepsilon|\le C\frac{R_\varepsilon^2}{\varepsilon^2}\frac{b_\varepsilon}{a_\varepsilon}$, we deduce

$$\begin{aligned}\int_{(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}}&|f^{\infty,p}(\nabla u_\varepsilon)-f^{\infty,p}(\nabla(\varphi_\varepsilon(x_3)u_\varepsilon(x)))|dx\le\int_{H_\varepsilon}|f^{\infty,p}(\nabla u_\varepsilon)|+|f^{\infty,p}(\nabla(\varphi_\varepsilon(x_3)u_\varepsilon(x)))|dx\\&\le C\int_{H_\varepsilon}|\nabla u_\varepsilon|^p+\left|\frac{u_\varepsilon(x)}{b_\varepsilon}\right|^p dx\le C\frac{b_\varepsilon}{a_\varepsilon}+\frac{C}{b_\varepsilon^p}\left(\int_{H_\varepsilon}|u_\varepsilon(x)|^{p^*}\,dx\right)^{\frac{p}{p^*}}|H_\varepsilon|^{\left(1-\frac{p}{p^*}\right)}\\&\le C\frac{b_\varepsilon}{a_\varepsilon}+\frac{C}{b_\varepsilon^p}\left(\frac{R_\varepsilon^2}{\varepsilon^2}\frac{b_\varepsilon}{a_\varepsilon}\right)^{\frac{p}{3}}=C\frac{b_\varepsilon}{a_\varepsilon}+C\left(\frac{R_\varepsilon^2}{\varepsilon^2}\frac{1}{a_\varepsilon b_\varepsilon^2}\right)^{\frac{p}{3}}=o(1).\end{aligned} \tag{7.11}$$

Since $\varphi_\varepsilon=0$ on $\bigcup_{k=0}^{m_\varepsilon+1}\{l_{k,\varepsilon}\}$, by applying Jensen's inequality, we get (see (7.4))

$$\begin{aligned}\int_{(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}}& f^{\infty,p}(\nabla(\varphi_\varepsilon(x_3)u_\varepsilon(x)))\,dx=\int_{D_{R_\varepsilon}\setminus S_{r_\varepsilon}}d\widehat{x}\sum_{k=1}^{m_\varepsilon+1}\int_{(l_{k-1,\varepsilon};l_{k,\varepsilon})}f^{\infty,p}(\nabla(\varphi_\varepsilon(x_3)u_\varepsilon(x)))\,dx_3\\&\ge\int_{D_{R_\varepsilon}\setminus S_{r_\varepsilon}}d\widehat{x}\sum_{k=1}^{m_\varepsilon+1}(l_{k,\varepsilon}-l_{k-1,\varepsilon})f^{\infty,p}\left(⨍_{(l_{k-1,\varepsilon};l_{k,\varepsilon})}\nabla(\varphi_\varepsilon(x_3)u_\varepsilon(x))\,dx_3\right)\\&=\int_{D_{R_\varepsilon}\setminus S_{r_\varepsilon}}d\widehat{x}\sum_{k=1}^{m_\varepsilon+1}(l_{k,\varepsilon}-l_{k-1,\varepsilon})f^{\infty,p}\left(\widehat{\nabla}\left(⨍_{(l_{k-1,\varepsilon};l_{k,\varepsilon})}\varphi_\varepsilon(x_3)u_\varepsilon(x)\,dx_3\right),0\right)\\&=\int_{(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}}f^{\infty,p}(\widehat{\nabla}\widehat{u}_\varepsilon,0)\,dx.\end{aligned} \tag{7.12}$$

The last line of (5.9) results from (7.9), (7.11) and (7.12).

□

The proof of the next Lemma relies on De Giorgi's slicing argument (see Remark 7.1).

**Lemma 7.2.** *Given a bounded sequence $(u_\varepsilon)$ in $W^{1,p}(\mathcal{O})$, there exists a sequence $(R_\varepsilon)$ satisfying (2.6) and*

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\int_{(D_{R_\varepsilon}\setminus D_{R_\varepsilon/2})\times(0,L)}|\nabla u_\varepsilon|^p\,dx=0,\\&R_\varepsilon\le\frac{R}{r}R'_\varepsilon\quad \text{if } p=2 \text{ and } 0<\gamma^{(2)}<+\infty \text{ (see (5.19))}.\end{aligned} \tag{7.13}$$

*Proof.* We fix a sequence of positive real numbers $(Q_\varepsilon)$ satisfying (see (2.1))

$$r_\varepsilon\ll Q_\varepsilon\ll\varepsilon,\qquad 1<<\gamma_\varepsilon^{(p)}(Q_\varepsilon)\quad (\text{respectively, } Q_\varepsilon\ll R'_\varepsilon\ \text{ if } p=2 \text{ and } 0<\gamma^{(2)}<+\infty).$$

(set for instance $Q_\varepsilon=\varepsilon^h$ with $1<h<\frac{2}{2-p}$ if $p<2$) and a sequence of positive integers $(q_\varepsilon)$ such that $\lim_{\varepsilon\to 0}q_\varepsilon=+\infty$, $r_\varepsilon\ll 2^{q_\varepsilon}Q_\varepsilon\ll\varepsilon$ (respectively, $r_\varepsilon\ll 2^{q_\varepsilon}Q_\varepsilon\ll R'_\varepsilon$ if $p=2$ and $0<\gamma^{(2)}<+\infty$). For each $\varepsilon>0$, the family of sets $(D_{2^mQ_\varepsilon}\setminus D_{2^{m-1}Q_\varepsilon})_{m\in\mathbb{N},1\le m\le q_\varepsilon}$ is disjoint, therefore

$$\sum_{m=1}^{q_\varepsilon}\int_{(D_{2^mQ_\varepsilon}\setminus D_{2^{m-1}Q_\varepsilon})\times(0,L)}|\nabla u_\varepsilon|^p\,dx\le\int_{\mathcal{O}}|\nabla u_\varepsilon|^p\,dx\le C.$$

Hence, for each $\varepsilon > 0$, there exists an integer $m_\varepsilon$ such that $1 \le m_\varepsilon \le q_\varepsilon$ and

$$\int_{(D_{2^{m_\varepsilon}Q_\varepsilon} \setminus D_{2^{m_\varepsilon-1}Q_\varepsilon})\times(0,L)} |\nabla u_\varepsilon|^p \, dx \le \frac{C}{q_\varepsilon}.$$

The sequence $(R_\varepsilon)$ defined by $R_\varepsilon = 2^{m_\varepsilon} Q_\varepsilon$ satisfies (2.6) and (7.13). $\square$

**Lemma 7.3.** *Assume that $(R_\varepsilon)$ satisfies (2.6) and (7.13). Then, there exists an approximation $\widehat{u}_\varepsilon$ of $\widehat{u}_\varepsilon$ verifying (5.13).*

*Proof.* The sequence $(\widehat{u}_\varepsilon)$ will be defined as follows: we choose a sequence $(R_\varepsilon)$ satisfying (2.6) and (7.13), fix $\zeta_\varepsilon, \xi_\varepsilon \in \mathcal{C}^\infty(\overline{\mathcal{O}})$ such that

$$\begin{aligned} &\zeta_\varepsilon = 0 \text{ in } D_{R_\varepsilon/2} \times (0,L), \quad \zeta_\varepsilon = 1 \text{ on } \partial D_{R_\varepsilon} \times (0,L), \quad |\widehat{\nabla}\zeta_\varepsilon| \le \frac{C}{R_\varepsilon}, \\ &\xi_\varepsilon = 0 \text{ in } (D_{R_\varepsilon} \setminus S_{r_\varepsilon}) \times (0,L), \quad \xi_\varepsilon = 1 \text{ in } S_{r_\varepsilon}^{-r_\varepsilon^\delta} \times (0,L), \quad |\widehat{\nabla}\xi_\varepsilon| \le \frac{C}{r_\varepsilon^\delta}, \end{aligned} \tag{7.14}$$

where $S_{r_\varepsilon}^{-r_\varepsilon^\delta} := \bigcup_{j\in J_\varepsilon} S_{r_\varepsilon}^{j,-r_\varepsilon^\delta}$ (see (5.11)), and set

$$\widehat{u}_\varepsilon := \widehat{u}_\varepsilon + \zeta_\varepsilon(\langle \widehat{u}_\varepsilon\rangle_{R_\varepsilon} - \widehat{u}_\varepsilon) + \xi_\varepsilon(\langle \widehat{u}_\varepsilon\rangle_{r_\varepsilon} - \widehat{u}_\varepsilon). \tag{7.15}$$

Any convex function $h$ on $\mathbb{R}^3$ verifying (1.9) satisfies (see [13, Proposition 2.32])

$$\exists C > 0; \; |h(a) - h(b)| \le C|a-b|(1 + |a|^{p-1} + |b|^{p-1}) \quad \forall\, a, \; b \in \mathbb{R}^3,$$

and by Hölder inequality, for all measurable set $A \subset \mathbb{R}^3$ and all $\varphi, \; \varphi' \in L^p(A)$, there holds

$$\left| \int_A h(\varphi)\, dx - \int_A h(\varphi')\, dx \right| \le C|\varphi - \varphi'|_{L^p(A)}\big(|A|^{\frac{p-1}{p}} + |\varphi|^{p-1}_{L^p(A)} + |\varphi'|^{p-1}_{L^p(A)}\big). \tag{7.16}$$

Applying (7.16) we infer

$$\begin{aligned} &\left| \int_{(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}} f^{\infty,p}\big(\widehat{\nabla}(\widehat{u}_\varepsilon), 0\big) - f^{\infty,p}\big(\widehat{\nabla}(\widehat{u}_\varepsilon), 0\big)\, dx \right| \\ &\qquad \le |\widehat{\nabla}(\widehat{u}_\varepsilon - \widehat{u}_\varepsilon)|_{E_\varepsilon}\big(|E_\varepsilon|^{\frac{p-1}{p}} + |\widehat{\nabla}(\widehat{u}_\varepsilon)|^{p-1}_{E_\varepsilon} + |\widehat{\nabla}(\widehat{u}_\varepsilon)|^{p-1}_{E_\varepsilon}\big) \\ &\qquad \le C|\widehat{\nabla}(\widehat{u}_\varepsilon - \widehat{u}_\varepsilon)|_{E_\varepsilon}\big(|E_\varepsilon|^{\frac{p-1}{p}} + |\widehat{\nabla}(\widehat{u}_\varepsilon)|^{p-1}_{E_\varepsilon} + |\widehat{\nabla}(\widehat{u}_\varepsilon - \widehat{u}_\varepsilon)|^{p-1}_{E_\varepsilon}\big), \\ &E_\varepsilon := L^p((D_{R_\varepsilon} \times (0,L)) \setminus T_{r_\varepsilon}; \mathbb{R}^2). \end{aligned} \tag{7.17}$$

We deduce from (5.9), (7.14), (7.15) and from the next estimate (obtained in a similar way as the fifth estimate of (4.4))

$$\int_{(D_{R_\varepsilon}\setminus D_{R_\varepsilon/2})\times(0,L)} |\widehat{u}_\varepsilon - \langle \widehat{u}_\varepsilon\rangle_{R_\varepsilon}|^p \, dx \le C R_\varepsilon^p \int_{(D_{R_\varepsilon}\setminus D_{R_\varepsilon/2})\times(0,L)} |\widehat{\nabla}\widehat{u}_\varepsilon|^p \, dx,$$

that

$$\begin{aligned} \int_{(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}} |\widehat{\nabla}(\widehat{u}_\varepsilon - \widehat{u}_\varepsilon)|^p \, dx &\le C \int_{(D_{R_\varepsilon}\setminus D_{R_\varepsilon/2})\times(0,L)} |\widehat{\nabla}\widehat{u}_\varepsilon|^p + |\widehat{u}_\varepsilon - \langle \widehat{u}_\varepsilon\rangle_{R_\varepsilon}|^p / R_\varepsilon^p \, dx \\ &\le C \int_{(D_{R_\varepsilon}\setminus D_{R_\varepsilon/2})\times(0,L)} |\nabla u_\varepsilon|^p \, dx. \end{aligned} \tag{7.18}$$

By (7.13), (7.17), and (7.18), there holds

$$\int_{(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}} f^{\infty,p}\big(\widehat{\nabla}\widehat{u}_\varepsilon, 0\big) - f^{\infty,p}\big(\widehat{\nabla}\widehat{u}_\varepsilon, 0\big)\, dx = o(1). \tag{7.19}$$

On the other hand, by (1.9), the last line of (4.4), (5.9), (7.14), and (7.15), we have

$$\begin{aligned}\int_{(S_{r_\varepsilon}\setminus S_{r_\varepsilon}^{-r_\varepsilon^\delta})\times(0,L)} f^{\infty,p}(\widehat{\nabla}\widehat{u}_\varepsilon,0)\,dx &\leq C\int_{(S_{r_\varepsilon}\setminus S_{r_\varepsilon}^{-r_\varepsilon^\delta})\times(0,L)} \left(|\widehat{\nabla}\widehat{u}_\varepsilon|^p + |\widehat{u}_\varepsilon - \langle\widehat{u}_\varepsilon\rangle_{r_\varepsilon}|^p/r_\varepsilon^{\delta p}\right)\,dx\\ &\leq C\Big(\frac{1+r_\varepsilon^{p(1-\delta)}}{\lambda_\varepsilon}\Big)\lambda_\varepsilon\int_{T_{r_\varepsilon}}|\nabla u_\varepsilon|^p\,dx\\ &\leq \frac{Cr_\varepsilon^{2+p(1-\delta)}}{\varepsilon^2} = \frac{Cr_\varepsilon^{2-p}}{\varepsilon^2}r_\varepsilon^{p(2-\delta)} = o(1),\end{aligned} \tag{7.20}$$

because $\gamma^{(p)} < +\infty$ and $1<\delta<2$ (see (2.1), (5.10)). By (7.15), (7.19), and (7.20), the first line of (5.13) is proved. The second line of (5.13) follows from (4.2), (5.12), (7.14), (7.15). □

**Remark 7.1.** *De Giorgi's slicing argument [14] is based on the following observation: if for each $\varepsilon>0$, $(A_\varepsilon^i)_{i\in\{1,\dots,l_\varepsilon\}}$ denotes a family of disjoint $\mu$-measurable subsets of a set $A$ equipped with a measure $\mu$, and if $(f_\varepsilon)$ is a sequence in $L^1_\mu(A)$ such that $|f_\varepsilon|_{L^1_\mu(A)}\leq C$, then for each $\varepsilon>0$, there exists $i_\varepsilon\in\{1,\dots,l_\varepsilon\}$ such that $\int_{A_\varepsilon^{i_\varepsilon}}|f_\varepsilon|d\mu\leq\frac{C}{l_\varepsilon}$. This argument is especially useful when non uniformly integrable sequences bounded in $L^1_\mu$ are considered. We employ this argument in the proof of Lemma 7.1 to establish the existence of the set $H_\varepsilon$ satisfying (7.2) and in the proof of Lemma 7.2.*


# Acknowledgment

During this work, Somsak is supported by a research fund from Commission on Higher Education and the Thailand Research Fund (MRG5080422).


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