Variational methods applied to discrete models in brittle damage

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Abstract. In this talk, I will study a discrete model of brittle damage in different regimes where the damaged zone concentrates on vanishingly small sets. We will identify the nature of the effective limit models obtained by means of an asymptotic analysis based on the Γ-convergence of the total energies. I will recall the mechanical model of brittle damage introduced by Francfort and Marigo [5], specified to the discrete setting where the total energies are restricted to piecewise affine continuous displacements. More precisely, given ε and $\eta_\varepsilon > 0$, we consider a linearly elastic material, whose reference configuration is a bounded open set $\Omega \subset \mathbb{R}^2$, which is composed of only two phases: a damaged phase (where the elasticity of the medium is altered) and a sound one, whose elasticity properties are given by η_ε and 1 respectively. Introducing the characteristic function of the damaged region, $\chi \in L^\infty(\Omega; \{0,1\})$, Francfort-Marigo's model consists in defining the total energy associated to a displacement $u \in H^1(\Omega; \mathbb{R}^2)$ as the sum of the elastic energy stored inside the material and a dissipative energy, taken as proportional to the volume of the damaged zone:

$$\mathcal{E}_\varepsilon(u,\chi) = \frac{1}{2}\int_\Omega (\chi\eta_\varepsilon + (1-\chi))\,|e(u)|^2\,dx + \frac{\kappa}{\varepsilon}\int_\Omega \chi\,dx,$$

where $e(u) = \left(\nabla u + \nabla u^T\right)/2$ is the linearized elastic strain and $\kappa/\varepsilon > 0$ is the material's toughness in the damaged regions. Note that the elasticity coefficients η_ε of the weak material degenerate while the diverging character of κ/ε as $\varepsilon \searrow 0$ forces the damaged zones to concentrate on vanishingly small sets. Here, we consider the total energies restricted to couples $(u,\chi) \in \mathcal{C}^0(\Omega; \mathbb{R}^2) \times L^\infty(\Omega, \{0,1\})$ in the finite element set

$$(u,\chi) \in X_{h_\varepsilon}(\Omega),$$

for which there exists a triangulation $\mathbf{T}_{h_\varepsilon}$ of Ω, whose mesh-size is of order $h_\varepsilon > 0$, such that u is affine and χ is constant on each of its triangle. According to the convergence rates $\alpha = \lim_{\varepsilon \searrow 0} \frac{\eta_\varepsilon}{\varepsilon} \in [0,+\infty]$ and $\beta = \lim_{\varepsilon \searrow 0} \frac{h_\varepsilon}{\varepsilon} \in [0,+\infty]$, we will exhibit the following effective limit models (some of them are still in progress):

Regime	**Effective limit model**
$\alpha = +\infty$ or $\beta = +\infty$	linear elasticity (see [3])
$\alpha = \beta = 0$	trivial model (see [3])
$\alpha = 0$ and $\beta \in (0,+\infty)$	brittle fracture (see [2])
$\alpha \in (0,+\infty)$ and $\beta = 0$	Hencky plasticity
$\alpha, \beta \in (0,+\infty)$	in between plasticity and brittle fracture (see [1, 4])

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